Department of Mathematics, and Statistics
University of Calgary

Math 311

Sheet 8A

1. Give an example of each of the following:
   a. A subset, U, of a vector space, V, which is **not** a subspace of the vector space, V..
   b. A subset, B, of a vector space, V, which is not linearly independent.
   c. A subset, B, of a vector space, V, which is linearly independent.
   d. A mapping, T, between two vector spaces, U and V, which is a linear transformation and which is one-to-one.
   e. A mapping, T, between two vector spaces, V and W, which is a linear transformation and which is onto.
   f. A subset, B, of a vector space, V, consisting of at least three vectors and which is an orthogonal set and which is not an orthonormal set.
   g. A subset, B, of a vector space, V, consisting of at least three vectors and which is an orthonormal set but is not one of the standard bases of the vector space.
   h. A 3x3 matrix which is positive definite and which is not a diagonal matrix..
   i. A 3x3 matrix which is not positive definite and which is not a diagonal matrix..

2. Give a definition for each of the following:
   a. A basis of a vector space.
   b. Dimension of a vector space
   c. A subspace of a vector space.
   d. Rank of an mxn matrix A.
   e. A linear tranformation, T, between the vector spaces U and V.
   f. Ker T, where T is a linear transformation between vector spaces U and V.
   g. Im T, where T is a linear transformation between vector spaces U and V.
   h. A one-to-one linear transformation, T between the vector spaces U and V.
   i. A linear transformation, T, between the vector spaces U and V which is onto.
   j. An isomorphism T, between two vector spaces U and V..
   k. A linearly independent set of vectors.
   l. A linearly dependent set of vectors.
   m. An orthogonal set of vectors.
   n. An orthonormal set of vectors.
   o. An eigenvalue of an nxn matrix.
   p. An eigenvector of an nxn matrix.
   q. An orthogonal matrix, P.
   r. A symmetric matrix, C.
   s. An nxn matrix A is similar to an nxn matrix B.
   t. An nxn matrix, A, is diagonalizable.

Department of Mathematics, and Statistics
University of Calgary

Math 311

Sheet 8A

u. An nxn matrix, B, is orthogonally diagonalizable.
v. An nxn matrix, A, is positive definite.
w. The rank of a linear transformation T from the vector space U to the vector space V.

3. T is a linear transformation between the vector spaces U and V.
   a. Show that ker T = $\{\mathbf{0}\}$ if and only if T is one-to-one.
   b. Show that Im T = V if and only if T is onto.
   c. Show that if T is one-to-one then every set of linearly independent vectors in U is mapped to a set of linearly independent vectors in V.

4. In each case, either prove the statement or give an example in which it is false. Throughout, let T be a linear transformation between V and W where V and W are finite dimensional vector spaces.
   a. If V = W, then $ker\ T \subseteq Im\ T$.
   b. If dim V = 5, and dim W = 3, and dim(kerT) = 2, then T is onto.
   c. If dim V = 5, and dim W = 4, then $ker\ T \neq \{\mathbf{0}\}$
   d. If $ker\ T = V$, then $W = \{\mathbf{0}\}$
   e. If $W = \{\mathbf{0}\}$, then $ker\ T = V$.
   f. If W = V, and $Im\ T \subseteq Ker\ T$, then T~=~0.
   g. If V has the basis $\{e_1, e_2, e_3\}$, and if $T(e_1) = \mathbf{0} = T(e_2)$, then $dim\ (im\ T) \leq 1$.
   h. If T is one-to-one, then $dim\ V \leq dim\ W$
   i. If $dim\ V \leq dim\ W$, then T is one-to-one.
   j. If T is onto, then $dim\ V \geq dim\ W$.

Department of Mathematics, and Statistics
University of Calgary.

Math 311

Sheet 8A

k. If $dim\ V \geq dim\ W$, then T is onto.

l. If $dim\ (ker\ T) \leq dim\ W$, then $dim\ W \geq \frac{1}{2}\ dim\ V$.

5. Find a linear transformation with the given properties:

a. $T : \Re^2 \to \Re^3$ such that $T\ (1,2) = (1,0,1)$; $T\ (-1,0) = (0,1,1)$. Find $T\ (2,1)$.

b. $T : P_2 \to P_3$ such that $T\ (x^2) = x^3$; $T(x + 1) = 0$; $T(x - 1) = x$. Find $T\left(x^2 + x - 1\right)$.

6. $T : V \to W$ is a linear transformation between vector spaces V and W. Show that $T\left(v - v_1\right) = T\left(v\right) - T\left(v_1\right)\ \ for\ \ all\ \ v,\ v_1 \in V$

7. $T : V \to W$ is a linear transformation between vector spaces V and W. Show that

a. If U is a subspace of V, then $T\ (U) = \left\{ T(\overline{u}) : \overline{u} \in U \right\}$ is a subspace of W.

b. If P is a subspace of W, then $T^{-1}\ (\ P\ ) = \left\{ v \in V : T\left(\ \overline{v}\ \right) \in P \right\}$ is a subspace of V.

8. Let $T : \Re^m \to \Re^n$ be a linear transformation with vectors written in rows.

a. Show that there exists an mxn matrix A such that $T\left(\ \overline{x}\ \right) = \overline{x}\ A\ \ for\ \ all\ \ \overline{x} \in \Re^m$

b. If $\left\{ \overline{e_1}, \ldots \overline{e_m} \right\}$ is the standard basis for $\Re^m$, show that the rows of A are

$$T\left(\ \overline{e_1}\ \right), T\left(\ \overline{e_2}\ \right), \ldots T\left(\ \overline{e_m}\ \right).$$

9. Let $T : V \to W$ be a linear transformation. Let $v_1, \ldots, v_n \in V$.

a. If $\left\{ T\left(\ \overline{v_i}\ \right) \right\}_{i=1}^{n}$ is linearly independent, show that $\left\{ \overline{v_i} \right\}_{i=1}^{n}$ is linearly independent.

Sheet 8A

b. Give an example to demonstrate that the converse of (a) is false.

10. If $T : V \to V$ is a linear transformation (usually such a linear transformation is called a linear operator) such that $T\left(T\left(\bar{v}\right)\right) = \bar{v},\ for\ all\ \bar{v} \in V$. If $\mathbf{0} \neq \bar{v} \in V$, show that $\left\{\bar{v},\ T\left(\bar{v}\right)\right\}$ is linearly independent if and only if $T\left(\bar{v}\right) \neq \bar{v}$ and $T\left(\bar{v}\right) \neq -\bar{v}$.

11. For any $a \in \mathfrak{R}$, define the <u>evaluation map</u> $E_a : P_n \to \mathfrak{R}$ by
$E_a\left(p(x)\right) = p(a) \quad for \quad p(x) \in P_n$.

a. Show that E_a is a linear transformation which satisfies the property
$E_a\left(x^k\right) = \left(E_a(x)\right)^k \quad for \quad k \in \{0,1,2,\dots\}$

b. If $T : P_n \to \mathfrak{R}$ is a linear transformation which satisfies
$T\left(x^k\right) = \left(T(x)\right)^k, \quad k \in \{0, 1, 2, \dots\}$, show that $T = E_a$ for some $a \in \mathfrak{R}$.

12. Find a basis of (1) Ker T, and (2) Im T in each case:

a. $T : P_2 \to \mathfrak{R}^2 \quad such\ that\ T(a + bx + cx^2) = (a, b)$

b. $T : \mathfrak{R}^3 \to \mathfrak{R}^3 \quad such\ that\ T(x, y, z) = (x + y, 0, x + y)$

c. $T : M_{22} \to M_{22} \quad such\ that\ T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & b+c \\ c+d & d+a \end{pmatrix}$

d. $T : M_{22} \to \mathfrak{R} \quad such\ that\ T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$

e. $T : M_{22} \to M_{22} \quad such\ that\ T(X) = XA - AX \quad where\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Department of Mathematics, and Statistics
University of Calgary

Math 311

Sheet 8A

13. Let $P : V \to \Re$ and $Q : V \to \Re$ be linear transformations. Let $T : V \to \Re^2$ be defined by $T(v) = (P(v), Q(v))$:

    a. Show that T is a linear tranformation.

    b. Show that $ker\ T = Ker\ P \cap Ker\ Q$

14. Use the Gram-Schmidt algorithm to convert the given basis of V into an orthogonal basis of V.

    a. $V = \mathbb{R}^3, \quad B = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

    b. $V = \mathbb{R}^4, \quad B = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ 2 \end{pmatrix} \right\}$ is a basis for a subspace of $\mathbb{R}^4$ and must first be extended to a basis for $\mathbb{R}^4$

15. If U is a subspace of a vector space, $\mathbb{R}^n$, show that

    a. $U^{\perp}$ is a subspace of $\mathbb{R}^n$.

    b. $U^{\perp\perp} = U$.