Sheet 6A
Math 311-L20

Spring 2009

1. Find a basis for $\mathbb{R}^3$ which contains the vector $v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

2. Let A be a 3x3 matrix with columns C_1, C_2, C_3. None of the columns of A are zero.
Show that
   a. $C_i^T C_i \neq 0 \qquad 1 \leq i \leq 3$
   b. $C_i^T C_j = C_j^T C_i \qquad 1 \leq i,j \leq 3$
   c. If $C_1^T C_2 = C_2^T C_3 = C_3^T C_1 = 0$ then $\{C_1, C_2, C_3\}$ is a linearly independent set.

3. U, V, and W are vector spaces. T and S are linear transformations between V and W, and W and U respectively [$T : V \to W; \quad S : W \to U$]. Show that $ker\ T \subseteq ker\ ST$.

4. Definition: Let x and y be vectors in $\mathbb{R}^n$, x and y are orthogonal if and only if $x \cdot y = 0$,
Problem:
If x and y are vectors in $\mathbb{R}^n$, show that x and y are orthogonal if and only if $\| x + y \| = \| x - y \|$.

5. For what real values of x will the following set of vectors in $\mathbb{R}^3$ be linearly independent?
$B = \{(2x, -1, -1), (-1, 2x, -1), (-1, -1, 2x)\}$

6. a. In the vector space $\mathbb{R}^3$, extend the given set to a basis for $\mathbb{R}^3$.
   $B = \{ (-1, 2, 3), (1, -2, -2) \}$
   b. Find a basis for $\mathbb{R}^4$ which contains the vectors
   $v = (1,-1,1,-1), \quad w = (0,1,0,1)$.

7. Consider the vector space M_{nn}. Let V be the set of all nxn matrices with the property that $A^T = -A$. Determine whether or not V is a subspace of M_{nn}. Justify your answer.

8. Let A be the 4x5 matrix, $A = \begin{pmatrix} 1 & -1 & 5 & -2 & 2 \\ 2 & -2 & -2 & 5 & 1 \\ 0 & 0 & -12 & 9 & -3 \\ -1 & 1 & 7 & -7 & 1 \end{pmatrix}$ Determine bases for Row A, Col A, Null A. Determine the rank of A.

9. If U and V are subspaces of V and dim U = 2, show that either $U \subseteq W$ or $dim\ (U \cap W) \leq 1$.

10. Given that the set of vectors, $\{\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}\}$ are a basis for the vector space V, determine whether or not the given sets are bases for the vector space V. In each case justify your answer.

a. $\{\boldsymbol{u} + \boldsymbol{v},\ \boldsymbol{v} + \boldsymbol{w},\ \boldsymbol{w} + \boldsymbol{u}\}$

b. $\{2\boldsymbol{u} + \boldsymbol{v} + 3\boldsymbol{w},\ 3\boldsymbol{u} + \boldsymbol{v} - \boldsymbol{w},\ \boldsymbol{u} - 4\boldsymbol{w}\}$

11. Let $B = \{A_1, A_2, \ldots A_n\} \subseteq M_{mn}$ and let $D = \{A_1^T, A_2^T, \ldots A_n^T\} \subseteq M_{nm}$. Show that

a. B is linearly independent if and only if D is linearly independent.

b. B spans M_{mn} if and only if D spans M_{nm}.