MATHEMATICS 311 L02 WINTER 2006
ADDITIONAL EXERCISES

1. Find a basis for *null* A, *col* A and *row* A for each of the following matrices:

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix}$.

(b) $A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 1 & 2 \end{bmatrix}$.

2. Determine whether the following statements are true or false.
(a) If $\{X, Y, Z\}$ is a basis of $\mathbb{R}^3$ then $\{X+Y, Y+Z, Z+X\}$ is also a basis of $\mathbb{R}^3$.
(b) There exists a matrix A so that *null* $A \cap$ *row* $A \neq \{0\}$.
(c) If $X, Y \in \mathbb{R}^n$ then $\{X+Y, X-3Y, 2X+Y\}$ is linearly dependent.
(d) If $\mathcal{F}$ is an ordered basis of $\mathbb{R}^n$ then the $\mathcal{F}$-matrix of the identity operator $1_{\mathbb{R}^n}$ is the $n \times n$ identity matrix I.
(e) If $\mathcal{F}$ is an ordered basis of $\mathbb{R}^n$ and the $\mathcal{F}$-matrix of a linear operator T on $\mathbb{R}^n$ is the $n \times n$ identity matrix I then $T = 1_{\mathbb{R}^n}$.
(f) If $\mathcal{F}$ and $\mathcal{G}$ are an ordered bases of $\mathbb{R}^n$ and $c_{\mathbb{F}}(X) = c_{\mathbb{F}}(X)$ for some $X \in \mathbb{R}^n$ then $\mathcal{F} = \mathcal{G}$.
(g) If $\mathcal{F}$ is an ordered basis of $\mathbb{R}^n$ then $c_{\mathbb{F}}$ is an invertible operator on $\mathbb{R}^n$.
(h) If T is a linear operator on $\mathbb{R}^n$ and $\mathcal{F}$ and $\mathcal{G}$ are an ordered bases of $\mathbb{R}^n$ then $\det(M_{\mathcal{F}}(T)) = \det(M_{\mathcal{G}}(T))$.
(i) There exists an operator T on $\mathbb{R}^2$ so that $M_{\mathcal{F}}(T) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $M_{\mathcal{G}}(T) = \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix}$ for some bases $\mathcal{F}$ and $\mathcal{G}$ of $\mathbb{R}^2$.

3. Determine whether each of the following sets are linearly independent.
(a) $\left\{[1,1,1]^T,\ [1,-1,1]^T,\ [1,-2,4]^T\right\}$.
(b) $\left\{[-1,3,1]^T,\ [1,1,1]^T,\ [3,1,2]^T\right\}$.

4. Let $\mathcal{M} = \left\{[-1,3,1]^T,\ [1,1,1]^T,\ [3,1,2]^T, [1,5,2]^T\right\}$
(a) Verify that $\mathcal{M}$ spans $\mathbb{R}^3$.
(b) Find a subset of M that is a basis of $\mathbb{R}^3$.

5. Let $U = \left\{[x,y,z]^T \in \mathbb{R}^3 \mid x+y+z=0\right\}$ (in other words, U is the plane through the origin with normal $N = [1,1,1]^T$)
(a) Show that U is a subspace of $\mathbb{R}^3$.
(b) Find a basis of U.
(c) Verify that $[1,-2,1]^T \in U$ and find a basis of U containing $[1,-2,1]^T$.

6. Let $\mathcal{F} = \{F_1, F_2\}$ and $\mathcal{G} = \{G_1, G_2\}$ where $F_1 = [2,1]^T$, $F_2 = [3,2]^T$, $G_1 = [-2,1]^T$ and $G_2 = [3,-1]^T$.

(a) Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^2$. Find $P_{\mathcal{F}\leftarrow\mathcal{E}}$, and $P_{\mathcal{G}\leftarrow\mathcal{E}}$.
(b) Let $X = [1,1]^T$. Find $C_{\mathcal{F}}(X)$ and $C_{\mathcal{G}}(X)$.
(c) Given that $C_{\mathcal{F}}(Y) = [1,-2]^T$. Find Y and $C_{\mathcal{G}}(Y)$.
(d) Find $P_{\mathcal{G}\leftarrow\mathcal{F}}$.
(e) Let T be a linear operator on $\mathbb{R}^2$ so that $M_{\mathcal{F}}[T] = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Find the standard matrix of T.
(f) Is the operator T in part (e) invertible? If so, find $M_{\mathcal{F}}[T^{-1}]$.
(g) For the operator T in part (e), find $M_{\mathcal{G}}[T]$.
(h) Let S be the linear operator on $\mathbb{R}^2$ so that $M_{\mathcal{F}}[S] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Let T be the linear operator on $\mathbb{R}^2$ in part (e). Find $M_{\mathcal{G}}[S]$, $M_{\mathcal{F}}[T \circ S]$ and $M_{\mathcal{F}}[S \circ T]$