MATHEMATICS 311 L02 WINTER 2006
QUIZ 1 SOLUTION

Tuesday, January 24, 2006 **Duration: 30 minutes.**

1. Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$. Find $nullA$, and find a spanning set of $nullA$.

Solution: Note that $nullA = \{X \in \mathbb{R}^3 \mid AX = 0\}$. That means $nullA$ is the set of all solutions of the system $AX = 0$. Solving the system $AX = 0$, we get $X = t\,[-2,1,1]^T$. Thus, $nullA = \left\{t\,[-2,1,1]^T \mid t \in \mathbb{R}\right\} = span\left\{[-2,1,1]^T\right\}$ and so a spanning set for $nullA$ is $\left\{[-2,1,1]^T\right\}$.

2. Is the set $\left\{[1,2,3]^T, [4,5,6]^T, [7,8,9]^T\right\}$ linearly independent? Explain.

Solution: No, the set $\left\{[1,2,3]^T, [4,5,6]^T, [7,8,9]^T\right\}$ is not linearly independent because $[1,2,3]^T - 2\,[4,5,6]^T + [7,8,9]^T = [0,0,0]^T$.

3. In this question, X, Y, Z are vectors in $\mathbb{R}^n$. Prove or disprove each of the following statements **using only the definitions** of subspaces $\mathbb{R}^n$, and of linearly independence (dependence):

(a) If U is a subspace of $\mathbb{R}^n$, and both $2Y$ and $X - Y$ are in U then X and Y are in U.

Solution: This statement is true and here is a proof. Suppose that U is a subspace of $\mathbb{R}^n$, and both $2Y$ and $X - Y$ are in U. Since $2Y \in U$ and U is closed under scalar multiplication, $Y = \frac{1}{2}(2Y) \in U$. Now, since $X - Y \in U$ and $Y \in U$, and U is closed under vector addition, we have $X = (X - Y) + Y \in U$.

(b) If $\{X, Y\}$ is linearly independent then $\{X + Y,\ 2X - Y\}$ is linearly independent.

Solution: This statement is true and here is a proof. Suppose that $\{X, Y\}$ is linearly independent. We prove that $\{X + Y,\ 2X - Y\}$ is linearly independent. Suppose that $a\,(X + Y) + b\,(X - Y)$ for some $a.b \in \mathbb{R}$. This is equivalent to $(a + b)\,X + (a - b)\,Y = 0$, and so from the independence of $\{X, Y\}$, we get $a + b = 0$ and $a - b = 0$. it follows that $b = a = \frac{1}{2}\left[(a + b) + (a - b)\right] = 0$. Thus, $\{X + Y,\ 2X - Y\}$ is linearly independent.

Thursday, January 26, 2006 **Duration: 30 minutes.**

1. Let $A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Find $nullA$, and find a spanning set of $nullA$.

Solution: Note that $nullA = \{X \in \mathbb{R}^3 \mid AX = 0\}$. That means $nullA$ is the set of all solutions of the system $AX = 0$. Solving the system $AX = 0$, we get $X = s\,[-2,1,1,0]^T + t\,[0,-1,0,1]^T$. Thus, $nullA = \left\{s\,[-2,1,1,0]^T + t\,[0,-1,0,1]^T \mid s,t \in \mathbb{R}\right\} = span\left\{[-2,1,1,0]^T, [0,-1,0,1]^T\right.$ and so a spanning set for $nullA$ is $\left\{[-2,1,1,0]^T, [0,-1,0,1]^T\right\}$.

2. Is the set $\left\{[1,2,3]^T, [-1,1,2]^T, [-1,4,7]^T\right\}$ linearly independent? Explain.

Solution: No, the set $\left\{[1,2,3]^T, [-1,1,2]^T, [-1,4,7]^T\right\}$ is not linearly independent because $[1,2,3]^T + 2[-1,1,2]^T - [-1,4,7]^T = [0,0,0]^T$.

3. Prove each of the following statements:
(a) If $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ are non-zero vectors in $\mathbb{R}^3$ such that $\overrightarrow{u} \cdot \overrightarrow{v} = \overrightarrow{u} \cdot \overrightarrow{w} = \overrightarrow{v} \cdot \overrightarrow{w} = \overrightarrow{0}$ then $\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\}$ is linearly independent.
Solution: Suppose that $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ are non-zero vectors in $\mathbb{R}^3$ such that $\overrightarrow{u} \cdot \overrightarrow{v} = \overrightarrow{u} \cdot \overrightarrow{w} = \overrightarrow{v} \cdot \overrightarrow{w} = 0$. We prove that $\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\}$ is linearly independent. Suppose that $a\overrightarrow{u} + b\overrightarrow{v} + c\overrightarrow{w} = \overrightarrow{0}$. Then $\overrightarrow{u} \cdot (a\overrightarrow{u} + b\overrightarrow{v} + c\overrightarrow{w}) = \overrightarrow{u} \cdot \overrightarrow{0} = 0$, that is, $\overrightarrow{u} \cdot (a\overrightarrow{u}) + \overrightarrow{u} \cdot (b\overrightarrow{v}) + \overrightarrow{u} \cdot (c\overrightarrow{w}) = a(\overrightarrow{u} \cdot \overrightarrow{u}) + b(\overrightarrow{u} \cdot \overrightarrow{v}) + c(\overrightarrow{u} \cdot \overrightarrow{w}) = a\left\|\overrightarrow{u}\right\|^2 = 0$ (because $\overrightarrow{u} \cdot \overrightarrow{v} = \overrightarrow{u} \cdot \overrightarrow{w} = 0$). Since $\overrightarrow{u} \neq \overrightarrow{0}$ and $a\left\|\overrightarrow{u}\right\|^2 = 0$, we conclude that $a = 0$. Similarly, $b\left\|\overrightarrow{v}\right\|^2 = \overrightarrow{v} \cdot (a\overrightarrow{u} + b\overrightarrow{v} + c\overrightarrow{w}) = 0$, and $c\left\|\overrightarrow{v}\right\|^2 = \overrightarrow{w} \cdot (a\overrightarrow{u} + b\overrightarrow{v} + c\overrightarrow{w}) = 0$ where $\overrightarrow{v}, \overrightarrow{w}$ are non-zero vectors, and so $b = c = 0$ also. Thus, $\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\}$ is linearly independent.

(b) If A, B are $n \times n$ matrices then $null\ A \subseteq null\ (BA)$.
Solution: Suppose that A, B are $n \times n$ matrices. We prove that $null\ A \subseteq null\ (BA)$. Let $X \in null\ A$. Then $AX = 0$, and so $(BA)X = B(AX) = B0 = 0$, that is, $X \in null\ (BA)$. Thus, $null\ A \subseteq null\ (BA)$.