MATHEMATICS 311 L02 WINTER 2006
QUIZ 2 SOLUTION
Tuesday, February 7, 2006

1. Let $A = \begin{bmatrix} 2 & 1 & 3 & 1 & 3 \\ 1 & 1 & 1 & 1 & 2 \\ -3 & -2 & -4 & -2 & -5 \end{bmatrix}$. Find a basis for each of the three following subspaces: $nullA$, $rowA$ and $colA$.

Solution:

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 & 3 \\ 1 & 1 & 1 & 1 & 2 \\ -3 & -2 & -4 & -2 & -5 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \rightarrow \\ R_3 + 3R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{matrix} \rightarrow \\ R_2 - R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 \end{bmatrix} \begin{matrix} \rightarrow \\ R_3 - R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Solving the system $AX = 0$, we get

$nullA = \left\{ r\,[-2,1,1,0,0]^T + s\,[0,1,0,1,0]^T + t\,[-1,-1,0,0,1]^T \mid r,s,t \in \mathbb{R} \right\}$.

Thus, a basis of *null* A is $\left\{ [-2,1,1,0,0]^T, [0,1,0,1,0]^T, [-1,-1,0,0,1]^T \right\}$,

a basis of *row* A is $\left\{ [1,0,2,0,1]^T, [0,1,-1,1,1]^T \right\}$, and

a basis of *col* A is $\left\{ [2,1,-3]^T, [1,1,-1]^T \right\}$.

2. Prove that if $\{X, Y, Z, W\}$ is a basis of $\mathbb{R}^4$ then $\{X + W, Y, Z, W\}$ is also a basis of $\mathbb{R}^4$.
Solution: Suppose that $\{X, Y, Z, W\}$ is a basis of $\mathbb{R}^4$. Let $A = [X\ Y\ Z\ W]$ be the matrix with colums X, Y, Z, W, and $A = [X + W\ Y\ Z\ W]$ be the matrix with colums $X + W, Y, Z, W$. Since $\{X, Y, Z, W\}$ is a basis of $\mathbb{R}^4$, A is invertible and so $\det A \neq 0$. Now, from B we can get A by doing the elementary column operation $C_1 - C_4$, and thus, $\det B = \det A \neq 0$. Hence, B is invertible and so $\{X + W, Y, Z, W\}$ is also a basis of $\mathbb{R}^4$.

3. Disprove the statement: "If $\{X_1, X_2, X_3, X_4\}$ and $\{Y_1, Y_2, Y_3, Y_4\}$ are bases of $\mathbb{R}^4$ then $\{X_1 + Y_1, X_2 + Y_2, X_3 + Y_3, X_4 + Y_4\}$ is a basis of $\mathbb{R}^4$."
Solution: the above statement is false because $\{X_1, X_2, X_3, X_4\} = \{E_1, E_2, E_3, E_4\}$ and $\{Y_1, Y_2, Y_3, Y_4\} = \{-E_1, -E_2, -E_3, -E_4\}$ are bases of $\mathbb{R}^4$ but $\{0, 0, 0, 0\}$ is not a basis of $\mathbb{R}^4$ because it is not linearly independent.

QUIZ 2
Thursday, February 9, 2006

1. Let $A = \begin{bmatrix} 2 & 1 & -1 & 2 & 1 \\ 1 & 1 & 2 & 3 & -1 \\ -3 & -2 & -1 & -5 & 0 \end{bmatrix}$. Find a basis for each of the three following subspaces: $nullA$, $rowA$ and $colA$.
Solution:

$$A=\begin{bmatrix}2&1&-1&2&1\\1&1&2&3&-1\\-3&-2&-1&-5&0\end{bmatrix}\begin{matrix}R_1-R_2\\ \longrightarrow\\ R_3+3R_2\end{matrix}\begin{bmatrix}1&0&-3&-1&2\\1&1&2&3&-1\\0&1&5&4&-3\end{bmatrix}$$

$$\begin{matrix}\longrightarrow\\ R_2-R_1\end{matrix}\begin{bmatrix}1&0&-3&-1&2\\0&1&5&4&-3\\0&1&5&4&-3\end{bmatrix}\begin{matrix}\longrightarrow\\ R_3-R_2\end{matrix}\left[\begin{bmatrix}1&0&-3&-1&2\\0&1&5&4&-3\\0&0&0&0&0\end{bmatrix}\right]=B$$

Solving the system $AX=0$, we get
$nullA=\left\{r\,[3,-5,1,0,0]^T+s\,[1,-4,0,1,0]^T+t\,[-2,3,0,0,1]^T \mid r,s,t\in\mathbb{R}\right\}$.
Thus, a basis of *null* A is $\left\{[3,-5,1,0,0]^T,[1,-4,0,1,0]^T,[-2,3,0,0,1]^T\right\}$,
a basis of *row* A is $\left\{[1,0,-3,-1,2]^T,[0,1,5,4,-3]^T\right\}$, and
a basis of *col* A is $\left\{[2,1,-3]^T,[1,1,-2]^T\right\}$.

2. Prove that if $\{X,Y,Z\}$ is a basis of $\mathbb{R}^3$ and A is an invertible 3×3 matrix then $\{AX,AY,AZ\}$ is also a basis of $\mathbb{R}^3$.
Solution: Suppose that $\{X,Y,Z\}$ is a basis of $\mathbb{R}^3$. Let $B=[X\ Y\ Z]$ be the matrix with colums X,Y,Z, and $C=[AX\ AY\ AZ]=AB$ be the matrix with colums AX, AY, AZ. Since $\{X,Y,Z\}$ is a basis of $\mathbb{R}^3$, B is invertible. Since A and B are invertible, AB is invertible and so $\{AX,AY,AZ\}$ is also a basis of $\mathbb{R}^3$.

3. Disprove the statement: "If A and B are 3×3 matrices and $rankA=rankB=2$ then $rank\,(A+B)=2$.

Solution: the above statement is false because when $A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ and $B=-A$, we see that $rankA=rankB=2$ but $rank\,(A+B)=0\neq 2$.