MATHEMATICS 311 L02 WINTER 2006
QUIZ 3 SOLUTIONS
Tuesday, February 28, 2006

1. In each of the following, prove or disprove that U is a subspace of $\mathbb{M}_{2,2}$.

(a) $U = \{A \in \mathbb{M}_{2,2} \mid A^2 = A\}$.

Solution: U is not a subsapce of $\mathbb{M}_{2,2}$ because U is not closed under addition. This is because $I \in U$ (because $I^2 = I$) but $I + I = 2I \notin U$ because $(2I)^2 = 4I \neq 2I$.

(b) $U = \left\{A \in \mathbb{M}_{2,2} \mid A = -A^T\right\}$.

Solution: U is a subsapce of $\mathbb{M}_{2,2}$ and here is a proof. First, we note that $0 \in U$ because $0 = -0^T$. Next, we prove that U is closed under addition. Let $A, B \in U$. Then $A = -A^T$ and $B = -B^T$, and so $A + B = -A^T - B^T = -\left(A^T + B^T\right) = -(A+B)^T$ and hence $A + B \in U$. Lastly, we prove that U is closed under scalar multiplication. Let $A \in U$ and $k \in \mathbb{R}$. Since $A \in U$ we have $A = -A^T$ and so $kA = k\left(-A^T\right) = -(kA)^T$ which implies that $kA \in U$. Thus, U is a subsapce of $\mathbb{M}_{2,2}$.

2. Let $\mathcal{F} = \{F_1, F_2\}$ and $\mathcal{G} = \{G_1, G_2\}$ be ordered bases of $\mathbb{R}^2$, where $F_1 = [1,2]^T$, $F_2 = [2,3]^T$, $G_1 = [1,1]^T$, $G_2 = [2,1]^T$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator such that $M_{\mathcal{F}}(T) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Let $\mathcal{E} = \{E_1, E_2\}$ be the standard basis of $\mathbb{R}^2$.

(a) Find the transition matrices $P_{\mathcal{E}\leftarrow\mathcal{F}}$, $P_{\mathcal{F}\leftarrow\mathcal{E}}$, $P_{\mathcal{E}\leftarrow\mathcal{G}}$, and $P_{\mathcal{G}\leftarrow\mathcal{E}}$.

Solution:

$$P_{\mathcal{E}\leftarrow\mathcal{F}} = [F_1, F_2] = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix},$$

$$P_{\mathcal{F}\leftarrow\mathcal{E}} = P_{\mathcal{E}\leftarrow\mathcal{F}}^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix},$$

$$P_{\mathcal{E}\leftarrow\mathcal{G}} = [G_1, G_2] = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \text{ and}$$

$$P_{\mathcal{G}\leftarrow\mathcal{E}} = P_{\mathcal{E}\leftarrow\mathcal{G}}^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

(b) Find the standard matrix of T.

Solution: The standard matrix of T is $M_{\mathcal{E}}(T) = P_{\mathcal{E}\leftarrow\mathcal{F}} M_{\mathcal{F}}(T) P_{\mathcal{F}\leftarrow\mathcal{E}} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -7 & 6 \\ -10 & 9 \end{bmatrix}$

(c) Find $C_{\mathcal{G}}(T(E_1))$.

Solution: From part (b), we know that $T(E_1) = \begin{bmatrix} -7 \\ -10 \end{bmatrix}$, and so $C_{\mathcal{G}}(T(E_1)) = P_{\mathcal{G}\leftarrow\mathcal{E}} C_{\mathcal{E}}(T(E_1)) = P_{\mathcal{G}\leftarrow\mathcal{E}} T(E_1) = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -7 \\ -10 \end{bmatrix} = \begin{bmatrix} -13 \\ -3 \end{bmatrix}$

MATHEMATICS 311 L02 WINTER 2006
QUIZ 3 Thursday, March 2, 2006

1. In each of the following, prove or disprove that U is a subspace of .$\mathbb{M}_{2,2}$.
(a) $U = \{A \in \mathbb{M}_{2,2} \mid A \text{ is not invertible}\}$.
Solution: U is not a subsapce of $\mathbb{M}_{2,2}$ because U is not closed under addition. This is because $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in U$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in U$ but $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin U$.

(b) $U = \{A \in \mathbb{M}_{2,2} \mid AB = BA\}$ for some fixed matrix $B \in \mathbb{M}_{2,2}$.
Solution: U is a subsapce of $\mathbb{M}_{2,2}$ and here is a proof. First, we note that $0 \in U$ because $0B = B0$. Next, we prove that U is closed under addition. Let $M, N \in U$. Then $MB = BM$ and $NB = BN$, and so $(M + N) B = MB + NB = BM + BN = B (M + N)$, which means $M+N \in U$, Lastly, we prove that U is closed under scalar multiplication. Let $A \in U$ and $k \in \mathbb{R}$. Since $A \in U$ we have $AB = BA$ and so $(kA) B = k (AB) = k (BA) = B (kA)$ which implies that $kA \in U$. Thus, U is a subsapce of $\mathbb{M}_{2,2}$.

2. Let $\mathcal{F} = \{F_1, F_2\}$ and $\mathcal{G} = \{G_1, G_2\}$ be ordered bases of $\mathbb{R}^2$, where $F_1 = [1, 2]^T$, $F_2 = [2, 3]^T$, $G_1 = [1, 1]^T$, $G_2 = [2, 1]^T$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator whose standard matrix is $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Let $\mathcal{E} = \{E_1, E_2\}$ be the standard basis of $\mathbb{R}^2$.
(a) Find the transition matrices $P_{\mathcal{E}\leftarrow\mathcal{F}}$, $P_{\mathcal{F}\leftarrow\mathcal{E}}$, $P_{\mathcal{E}\leftarrow\mathcal{G}}$, and $P_{\mathcal{G}\leftarrow\mathcal{F}}$.
Solution:

$$P_{\mathcal{E}\leftarrow\mathcal{F}} = [F_1, F_2] = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix},$$

$$P_{\mathcal{F}\leftarrow\mathcal{E}} = P_{\mathcal{E}\leftarrow\mathcal{F}}^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix},$$

$$P_{\mathcal{E}\leftarrow\mathcal{G}} = [G_1, G_2] = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},$$

$$P_{\mathcal{G}\leftarrow\mathcal{E}} = P_{\mathcal{E}\leftarrow\mathcal{G}}^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}, \text{ and so}$$

$$P_{\mathcal{G}\leftarrow\mathcal{F}} = P_{\mathcal{G}\leftarrow\mathcal{E}} P_{\mathcal{E}\leftarrow\mathcal{F}} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$$

(b) Find $M_{\mathcal{F}}(T)$.
Solution: $M_{\mathcal{F}}(T) = P_{\mathcal{F}\leftarrow\mathcal{E}} M_{\mathcal{E}}(T) P_{\mathcal{E}\leftarrow\mathcal{F}} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -7 & -10 \\ 6 & 9 \end{bmatrix}$

(c) Find $C_{\mathcal{G}}(T(F_2))$.
Solution: From part (b), we know that $C_{\mathcal{F}}(T(F_2)) = \begin{bmatrix} -10 \\ 9 \end{bmatrix}$, and so $C_{\mathcal{G}}(T(F_2)) = P_{\mathcal{G}\leftarrow\mathcal{F}} C_{\mathcal{F}}(T(F_2)) = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -10 \\ 9 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$