MATHEMATICS 311 L02 WINTER 2006
QUIZ 4 SOLUTION
Tuesday, Mar 21, 2006

1. Consider the basis $\mathcal{B} = \{1,\ x + x^2,\ x^2 + 1\}$ of $\mathbb{P}_2$. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation so that $M_{\mathcal{B}\mathcal{B}}(T) = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

(a) Find $T(1)$, $T(x + x^2)$ and $T(x^2 + 1)$.

Hint: From $M_{\mathcal{B}\mathcal{B}}(T)$, you know the coordinates of $T(1)$, $T(x + x^2)$ and $T(x^2 + 1)$ in the basis $\mathcal{B}$. Use those.

Solution: From we know that $c_{\mathcal{B}}(T(1)) = [1, 1, 0]^T$, $c_{\mathcal{B}}(T(x + x^2)) = [1, 2, 0]^T$, and $c_{\mathcal{B}}(T(x^2 + 1)) = [-1, 1, -1]^T$. Thus,

$$\begin{aligned} T(1) &= 1 + (x + x^2) &&= 1 + x + x^2 \\ T(x + x^2) &= 1 + 2(x + x^2) &&= 1 + 2x + 2x^2 \\ T(x^2 + 1) &= -1 + (x + x^2) - (x^2 + 1) &&= -2 + x \end{aligned}$$

(b) Find $T(x^2)$ and $T(x)$.

Hint: $x^2 = (x^2 + 1) - 1$ and $x = (x + x^2) - x^2$, and use the linearity of T.

Solution: From $x^2 = (x^2 + 1) - 1$ and $x = (x + x^2) - x^2$, we get

$$\begin{aligned} T(x^2) &= T(x^2 + 1) - T(1) &&= (-2 + x) - (1 + x + x^2) &&= -3 - x^2 \\ T(x) &= T(x + x^2) - T(x^2) &&= (1 + 2x + 2x^2) - (-3 - x^2) &&= 4 + 2x + 3x^2 \end{aligned}$$

(c) Find $M_{\mathcal{E}\mathcal{E}}(T)$ where $\mathcal{E} = \{1,\ x,\ x^2\}$ is the standard basis of $\mathbb{P}_2$.

Solution:

$$M_{\mathcal{E}\mathcal{E}}(T) = [c_{\mathcal{E}}(T(1)), c_{\mathcal{E}}(T(x)), c_{\mathcal{E}}(T(x^2))] = \begin{bmatrix} 1 & 4 & -3 \\ 1 & 2 & 0 \\ 1 & 3 & -1 \end{bmatrix}$$

2. Let $T : V \to W$ be a linear transformation, let $\mathcal{B} = \left\{\overrightarrow{b_1}, \overrightarrow{b_2}, ..., \overrightarrow{b_k}\right\} \subseteq V$ and $\mathcal{C} = \left\{T\left(\overrightarrow{b_1}\right), T\left(\overrightarrow{b_2}\right), ..., T\left(\overrightarrow{b_k}\right)\right\}$.

(a) Prove that if $\mathcal{C}$ is linearly independent then $\mathcal{B}$ is linearly independent.

Solution: Suppose that $\mathcal{C}$ is linearly independent. We prove that $\mathcal{B}$ is linearly independent. Suppose that $a_1\overrightarrow{b_1} + a_2\overrightarrow{b_2} + ... + a_k\overrightarrow{b_k} = \overrightarrow{0}$ for some $a_1, a_2, ..., a_k \in \mathbb{R}$. Then $T\left(a_1\overrightarrow{b_1} + a_2\overrightarrow{b_2} + ... + a_k\overrightarrow{b_k}\right) = T\left(\overrightarrow{0}\right) = \overrightarrow{0}$, which implies that $a_1T\left(\overrightarrow{b_1}\right) + a_2T\left(\overrightarrow{b_2}\right) + ... + a_kT\left(\overrightarrow{b_k}\right) = \overrightarrow{0}$ and hence by the independence of $\mathcal{C}$, we get $a_1 = a_2 = ... = a_k = 0$. Thus, $\mathcal{B}$ is linearly independent.

(b) Prove that if T is one-to-one and $\mathcal{B}$ is linearly independent then $\mathcal{C}$ is linearly independent.

Solution: Suppose that T is one-to-one and that $\mathcal{B}$ is linearly independent. We prove that $\mathcal{C}$ is linearly independent. Suppose that $a_1T\left(\overrightarrow{b_1}\right) + a_2T\left(\overrightarrow{b_2}\right) + ... + a_kT\left(\overrightarrow{b_k}\right) = \overrightarrow{0}$ for some $a_1, a_2, ..., a_k \in \mathbb{R}$. Then $T\left(a_1\overrightarrow{b_1} + a_2\overrightarrow{b_2} + ... + a_k\overrightarrow{b_k}\right) = \overrightarrow{0} = T\left(\overrightarrow{0}\right)$, which implies that $a_1\overrightarrow{b_1} + a_2\overrightarrow{b_2} + ... + a_k\overrightarrow{b_k} = \overrightarrow{0}$ (because T is one-to-one)and hence by the independence of $\mathcal{B}$, we get $a_1 = a_2 = ... = a_k = 0$. Thus, $\mathcal{C}$ is linearly independent.

QUIZ 4 SOLUTION
Thursday, Mar 23, 2006

1. Consider the bases $\mathcal{B} = \{1+x,\ x+x^2,\ x^2+1\}$ and $\mathcal{E} = \{1,\ x,\ x^2\}$ of $\mathbb{P}_2$. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation so that $M_{\mathcal{B}\mathcal{E}}(T) = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

(a) Find $c_{\mathcal{B}}(T(1))$, $c_{\mathcal{B}}(T(x))$ and $c_{\mathcal{B}}(T(x^2))$.

Solution: From $M_{\mathcal{B}\mathcal{E}}(T)$ we know that $c_{\mathcal{B}}(T(1)) = [1,1,0]^T$, $c_{\mathcal{B}}(T(x)) = [1,2,0]^T$ and $c_{\mathcal{B}}(T(x^2)) = [-1,1,-1]^T$.

(b) Find $c_{\mathcal{B}}(T(1+x))$, $c_{\mathcal{B}}(T(x+x^2))$ and $c_{\mathcal{B}}(T(x^2+1))$.

Hint: $c_{\mathcal{B}}$ and T are linear transformations.

Solution:

$c_{\mathcal{B}}(T(1+x)) = c_{\mathcal{B}}(T(1)+T(x)) = c_{\mathcal{B}}(T(1)) + c_{\mathcal{B}}(T(x)) = [1,1,0]^T + [1,2,0]^T = [2,3,0]^T$.

$c_{\mathcal{B}}(T(x+x^2)) = c_{\mathcal{B}}(T(x)+T(x^2)) = c_{\mathcal{B}}(T(x)) + c_{\mathcal{B}}(T(x^2)) = [1,2,0]^T + [-1,1,-1]^T = [0,3,-1]^T$.

$c_{\mathcal{B}}(T(x^2+1)) = c_{\mathcal{B}}(T(x^2)+T(1)) = c_{\mathcal{B}}(T(x^2)) + c_{\mathcal{B}}(T(1)) = [-1,1,-1]^T + [1,1,0]^T = [0,2,-1]^T$.

(c) Find $M_{\mathcal{B}\mathcal{B}}(T)$.

Solution:

$$M_{\mathcal{B}\mathcal{B}}(T) = [c_{\mathcal{B}}(T(1+x)),\ c_{\mathcal{B}}(T(x+x^2)),\ c_{\mathcal{B}}(T(x^2+1))] = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 3 & 2 \\ 0 & -1 & -1 \end{bmatrix}$$

2. Let $T : V \to W$ be a linear transformation. We recall that $\ker T = \left\{ \overrightarrow{u} \in V \mid T(\overrightarrow{u}) = \overrightarrow{0} \right\}$ is a subspace of V. We recall the definition: T is one-to-one if and only if the statement: "if $\overrightarrow{u}, \overrightarrow{v} \in V$ and $T(\overrightarrow{u}) = T(\overrightarrow{v})$ then $\overrightarrow{u} = \overrightarrow{v}$."is true. Prove the following using the definition of $\ker T$ and definition of one-to-one.

(a) Prove that if $\ker T = \left\{ \overrightarrow{0} \right\}$ then T is one-to-one.

Solution: Suppose that $\ker T = \left\{ \overrightarrow{0} \right\}$, We prove that T is one-to-one. Suppose that $\overrightarrow{u}, \overrightarrow{v} \in V$ and $T(\overrightarrow{u}) = T(\overrightarrow{v})$. Then $T(\overrightarrow{u} - \overrightarrow{v}) = T(\overrightarrow{u}) - T(\overrightarrow{v}) = \overrightarrow{0}$. Thus, $\overrightarrow{u} - \overrightarrow{v} \in \ker T = \left\{ \overrightarrow{0} \right\}$. Hence, $\overrightarrow{u} - \overrightarrow{v} = \overrightarrow{0}$, and so $\overrightarrow{u} = \overrightarrow{v}$. Therefore, T is one-to-one.

(b) Prove that if T is one-to-one then $\ker T = \left\{ \overrightarrow{0} \right\}$.

Solution: Suppose that T is one-to-one. We prove that $\ker T = \left\{ \overrightarrow{0} \right\}$.

We note that since $T\left(\overrightarrow{0}\right) = \overrightarrow{0}$, $\overrightarrow{0} \in \ker T$ and so $\left\{ \overrightarrow{0} \right\} \subseteq \ker T$. Next, we prove that $\ker T \subseteq \left\{ \overrightarrow{0} \right\}$. Let $\overrightarrow{u} \in \ker T$, that is, $T(\overrightarrow{u}) = \overrightarrow{0} = T\left(\overrightarrow{0}\right)$, and since T is one-to-one, $\overrightarrow{u} = \overrightarrow{0} \in \left\{ \overrightarrow{0} \right\}$. Thus, $\ker T = \left\{ \overrightarrow{0} \right\}$.