MATHEMATICS 311 L02 WINTER 2005
MIDTERM SOLUTION

1. Write the definition of each of the following:
(a) A *subspace* of a vector space V.
Solution: A subspace of a vector space V is a subset U of V so that U is a vector space under the same vector addition, and scalar multimiplation in V.
(b) *Linearly independence* of a subset $\{\overrightarrow{v_1},\ \overrightarrow{v_2}, ...,\ \overrightarrow{v_k}\}$ of a vector space V.
Solution: A subset $\{\overrightarrow{v_1},\ \overrightarrow{v_2}, ...,\ \overrightarrow{v_k}\}$ of a vector space V is linearly indepentdent if and only if the following statement is true: "if $a_1\overrightarrow{v_1}+a_2\overrightarrow{v_2}+...+a_k\overrightarrow{v_k} = \overrightarrow{0}$ for some $a_1, a_2, ..., a_k \in \mathbb{R}$ then $a_1 = a_2 = ... = a_k = 0$."
(c) A *linear transformation* from $\mathbb{R}^n$ to $\mathbb{R}^m$.
Solution: A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ so that for all $X, Y \in \mathbb{R}^n$ and $k \in \mathbb{R}$, we have

(T1) $T(X+Y) = T(X) + T(Y)$, and

(T2) $T(kX) = kT(X)$.

(d) The $\mathcal{F}$*-coordinate vector* $C_{\mathcal{F}}(X)$ of a vector X in $\mathbb{R}^n$ where $\mathcal{F} = \{F_1,\ F_2, ...,\ F_n\}$ is an ordered basis of $\mathbb{R}^n$.
Solution: $C_{\mathcal{F}}(X) = [a_1, a_2, ..., a_n]^T$ if and only if $X = a_1F_1 + a_2F_2 + ... + a_nF_n$.
(e) The $\mathcal{F}$*-matrix* $M_{\mathcal{F}}(T)$ of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ where $\mathcal{F} = \{F_1,\ F_2, ...,\ F_n\}$ is an ordered basis of $\mathbb{R}^n$.
Solution: The $\mathcal{F}$*-matrix* $M_{\mathcal{F}}(T)$ of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the matrix so that $c_{\mathcal{F}}(T(X)) = M_{\mathcal{F}}(T)\, c_{\mathcal{F}}(X)$ for all $X \in \mathbb{R}^n$.

2. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & -1 \\ 3 & 2 & 4 & 6 \end{bmatrix}$. Recall that *null* $A = \{X \in \mathbb{R}^4 \mid AX = 0\}$ and *im* $A = \{AX \mid X \in \mathbb{R}^4\} =$ *col* A where *col* A is the subspace spanned by the columns of A.
(a) Find a basis of *null* A and a basis of *im* A.
Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & -1 \\ 3 & 2 & 4 & 6 \end{bmatrix} \begin{matrix} \to \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -3 \\ 0 & -1 & 1 & 3 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \to \\ R_3 + R_2 \end{matrix}$$
$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving the system $AX = 0$ we get $X = s[-2, 1, 1, 0]^T + t[-4, 3, 0, 1]^T$. Thus, a basis of *null* A is $\left\{[-2, 1, 1, 0]^T, [-4, 3, 0, 1]^T\right\}$ and a basis of *im* A is $\left\{[1, 2, 3]^T, [1, 3, 2]^T\right\}$
(b) Is $\left\{[-6, 4, 1, 1]^T, [2, -2, 1, -1]^T\right\}$ a basis of *null* A? Explain.
Solution: Yes, $\left\{[-6, 4, 1, 1]^T, [2, -2, 1, -1]^T\right\}$ is a basis of *null* A. Let $X_1 = [-6, 4, 1, 1]^T$ and $X_2 = [2, -2, 1, -1]^T$. We note that $AX_1 = AX_2 = 0$. Thus, $X_1, X_2 \in$ *null* A. Thus, since $\dim($*null* $A) = 2$ as seen in (a), to prove that $\{X_1, X_2\}$ is a basis of we only need to show that $\{X_1, X_2\}$ is linearly indepedent. Now, suppose $a[-6, 4, 1, 1]^T + b[2, -2, 1, -1]^T =$

$[0,0,0,0]^T$. Then we have $a+b=0$ and $a-b=0$, which implies $a=b$, and $a=\frac{1}{2}[(a+b)+(a-b)]=\frac{1}{2}[0+0]=0$, so $a=b=0$. Thus, $\left\{[-6,4,1,1]^T,[2,-2,1,-1]^T\right\}$ is a basis of *null* A.

3. Consider the basis $\mathcal{F}=\{F_1,F_2\}$ of $\mathbb{R}^2$ where $F_1=[1,-1]^T$, and $F_2=[-1,2]^T$. Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be a linear operator on $\mathbb{R}^2$ so that $T(F_1)=[2,1]^T$ and $T(F_2)=[3,2]^T$. Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^2$.
(a) Find the transition matrices $P_{\mathcal{E}\leftarrow\mathcal{F}}$ and $P_{\mathcal{F}\leftarrow\mathcal{E}}$.
Solution:
$P_{\mathcal{E}\leftarrow\mathcal{F}}=[F_1F_2]=\begin{bmatrix}1&-1\\-1&2\end{bmatrix}$ and $P_{\mathcal{F}\leftarrow\mathcal{E}}=\begin{bmatrix}1&-1\\-1&2\end{bmatrix}^{-1}=\begin{bmatrix}2&1\\1&1\end{bmatrix}$
(b) Find $C_{\mathcal{F}}(T(F_1))$ and $C_{\mathcal{F}}(T(F_2))$.
Solution:
$C_{\mathcal{F}}(T(F_1))=P_{\mathcal{F}\leftarrow\mathcal{E}}C_{\mathcal{E}}(T(F_1))=P_{\mathcal{F}\leftarrow\mathcal{E}}T(F_1)=\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}5\\3\end{bmatrix}$, and
$C_{\mathcal{F}}(T(F_2))=P_{\mathcal{F}\leftarrow\mathcal{E}}C_{\mathcal{E}}(T(F_2))=P_{\mathcal{F}\leftarrow\mathcal{E}}T(F_2)=\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}8\\5\end{bmatrix}$.
(c) Find $M_{\mathcal{F}}(T)$.
Solution:
$M_{\mathcal{F}}(T)=[C_{\mathcal{F}}(T(F_1)),C_{\mathcal{F}}(T(F_2))]=\begin{bmatrix}5&8\\3&5\end{bmatrix}$
(d) Find the standard matrix of T.
Solution:
The standard matrix of T is

$$\begin{aligned}M_{\mathcal{E}}(T)&=P_{\mathcal{E}\leftarrow\mathcal{F}}M_{\mathcal{F}}(T)P_{\mathcal{F}\leftarrow\mathcal{E}}\\&=\begin{bmatrix}1&-1\\-1&2\end{bmatrix}\begin{bmatrix}5&8\\3&5\end{bmatrix}\begin{bmatrix}2&1\\1&1\end{bmatrix}\\&=\begin{bmatrix}2&3\\1&2\end{bmatrix}\begin{bmatrix}2&1\\1&1\end{bmatrix}\\&=\begin{bmatrix}7&5\\4&3\end{bmatrix}\end{aligned}$$

4. Let V be a vector space. Let $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{w}$ be vectors in V.
(a) Prove that if $\{\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w}\}$ is a basis of V then $\{\overrightarrow{u}+\overrightarrow{v},\ \overrightarrow{v}+\overrightarrow{w},\ \overrightarrow{w}+\overrightarrow{u}\}$ is also a basis of V.
Solution: Suppose that $\{\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w}\}$ is a basis of V. Then $\dim V=3$, and so to prove that $\{\overrightarrow{u}+\overrightarrow{v},\ \overrightarrow{v}+\overrightarrow{w},\ \overrightarrow{w}+\overrightarrow{u}\}$ is also a basis of V, we only have to prove that $\{\overrightarrow{u}+\overrightarrow{v},\ \overrightarrow{v}+\overrightarrow{w},\ \overrightarrow{w}+\overrightarrow{u}\}$ is linearly independent. Now, suppose that $a(\overrightarrow{u}+\overrightarrow{v})+b(\overrightarrow{v}+\overrightarrow{w})+c(\overrightarrow{w}+\overrightarrow{u})=\overrightarrow{0}$, which is the same as $(a+c)\overrightarrow{u}+(a+b)\overrightarrow{v}+(b+c)\overrightarrow{w}=\overrightarrow{0}$, which, by the independence $\{\overrightarrow{u},\ \overrightarrow{v},\ \overrightarrow{w}\}$, implies that $a+c=a+b=b+c=0$. Thus, $a=\frac{1}{2}[(a+b)+(a+c)-(b+c)]=\frac{1}{2}[0+0+0]=0$. From $a+c=a+b=0$, we get $b=-a=0$ and $c=-a=0$. Thus, $a=b=c=0$, and so $\{\overrightarrow{u}+\overrightarrow{v},\ \overrightarrow{v}+\overrightarrow{w},\ \overrightarrow{w}+\overrightarrow{u}\}$ is also a basis of V.
(b) Prove that the set $\{\overrightarrow{u}+\overrightarrow{v},\ 2\overrightarrow{v}-3\overrightarrow{u},\ \overrightarrow{u}+4\overrightarrow{v}\}$ is linearly dependent.

Solution: We note that $\{\overrightarrow{u}+\overrightarrow{v},\ 2\overrightarrow{v}-3\overrightarrow{u},\ \overrightarrow{u}+4\overrightarrow{v}\}$ is a set of three vectors in the subspace of V spanned by two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$, therefore, by the Fundamental Theorem, $\{\overrightarrow{u}+\overrightarrow{v},\ 2\overrightarrow{v}-3\overrightarrow{u},\ \overrightarrow{u}+4\overrightarrow{v}\}$ is linearly dependent.