FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL EXAMINATION
MATH 311 (L 02) WINTER 2005 **Duration: 3 hours**

[10] 1. Let V and W be vector spaces. Write the definition of each of the following:

(a) A *basis* of a subspace U of V.

(b) *Linearly independence* of a finite subset $\{\overrightarrow{v}_1, \overrightarrow{v}_2, ..., \overrightarrow{v}_k\}$ of V.

(c) A *linear transformation* from V to W.

(d) An *orthonormal* set in $\mathbb{R}^n$.

(e) A *diagonalizable* matrix..

[10] 2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T\left([x, y, z]^T\right) = [y - 3z,\ -3y + 3z,\ -2y + 6z]^T$.

(a) Find a basis of $\ker T$, and find $\dim(\ker T)$.

(b) Find a basis of imT, and find $\dim(imT)$.

(c) Is T onto? Explain.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T\left([x, y, z]^T\right) = [x + y + z, x - y + z, x]^T$ for all $[x, y, z]^T \in \mathbb{R}^3$. Find the $\mathcal{F}$-matrix of T where $\mathcal{F} = \left\{[1, 1, 1]^T, [0, 1, 1]^T, [1, 0, 1]^T\right\}$ is a basis of $\mathbb{R}^3$.

[10] 4. Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^2$. Let $\mathcal{F} = \{F_1, F_2\}$ and $\mathcal{G} = \{G_1, G_2\}$ where $F_1 = [2, 1]^T$, $F_2 = [3, 2]^T$, $G_1 = [-2, 1]^T$ and $G_2 = [3, -1]^T$. Note that $\mathcal{F}$ and $\mathcal{G}$ are bases of $\mathbb{R}^2$. Let T be a linear operator on $\mathbb{R}^2$ with $M_{\mathcal{F}}[T] = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

(a) Find the standard matrix of T.

(b) Find $M_{\mathcal{G}}[T]$.

[10] 5. Let $\mathbb{M}_{2,2}$ be the vector space of all 2×2 matrices and let $T : \mathbb{M}_{2,2} \to \mathbb{M}_{2,2}$ be the linear transformation defined by $T(A) = A^T$ for all $A \in \mathbb{M}_{2,2}$. Find the matrix $M_{\mathcal{DB}}[T]$ where
$\mathcal{B} = \mathcal{D} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

[10] 6. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$. Find an orthogonal matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

[10] 7. Let $U = span\{X_1, X_2\}$ where $X_1 = [1, -1, 0, 1]^T$ and $X_2 = [1, 0, 1, 0]^T$. Let $X = X_1 + 2X_2$. Express X as the sum of a vector in U and a vector in $U^{\perp}$; that is, find vectors $P \in U$ and $Q \in U^{\perp}$ so that $X = P + Q$.

[10] 8. Let V be a vector space. Prove the following statements.

(a) For all vectors $\overrightarrow{u}, \overrightarrow{v} \in V$, if $\{\overrightarrow{u}, \overrightarrow{v}\}$ is linearly independent then $\{\overrightarrow{u} + \overrightarrow{v}, \overrightarrow{u} - \overrightarrow{v}\}$ is also linearly independent.

(b) For all vectors $\overrightarrow{u}, \overrightarrow{v} \in V$, $\{\overrightarrow{u} + \overrightarrow{v}, \overrightarrow{u} - \overrightarrow{v}, 2\overrightarrow{u} + 3\overrightarrow{v}\}$ is linearly dependent.

[10] 9. Let $T : V \to W$ be a linear transformation. Let $\mathcal{B} = \left\{\overrightarrow{b}_1, \overrightarrow{b}_2, ..., \overrightarrow{b}_n\right\}$ be a subset of V, and let $\mathcal{C} = \left\{T\left(\overrightarrow{b}_1\right), T\left(\overrightarrow{b}_2\right), ..., T\left(\overrightarrow{b}_n\right)\right\}$. Prove the following statements:

(a) If $\mathcal{C}$ is linearly independent then $\mathcal{B}$ is linearly independent.

(b) If $\mathcal{C}$ spans W then T is onto.

[10] 10. Let $\overrightarrow{v}_1,\ \overrightarrow{v}_2,\ ...,\ \overrightarrow{v}_n$ be vectors in a vector space V and define $T : \mathbb{R}^n \to V$ by $T\left([x_1,\ x_2, ...,\ x_n]^T\right) = x_1\overrightarrow{v}_1 + x_2\overrightarrow{v}_2 + ... + x_n\overrightarrow{v}_n$. Prove the following statements:

(a) If T is one-to-one then $\{\overrightarrow{v}_1,\ \overrightarrow{v}_2,\ ...,\ \overrightarrow{v}_n\}$ is linearly independent.

(b) If $\{\overrightarrow{v}_1,\ \overrightarrow{v}_2,\ ...,\ \overrightarrow{v}_n\}$ is linearly independent then T is one-to-one.

End of Examination