MATHEMATICS 311 L02 WINTER 2005
MIDTERM EXAMINATION
March 11, 2005 at 12:00 Duration: 50 minutes

[10] **1**. Write the definition of each of the following:
(a) A *subspace* of $\mathbb{R}^n$.
(b) *Linearly independence* of a finite subset $\{X_1,\ X_2, ...,\ X_k\}$ of $\mathbb{R}^n$.
(c) A *linear transformation* from $\mathbb{R}^n$ to $\mathbb{R}^m$.
(d) A *basis* of a subspace U of $\mathbb{R}^n$.
(e) The $\mathcal{F}$-*coordinate vector* $C_{\mathcal{F}}(X)$ of a vector X in $\mathbb{R}^n$ where $\mathcal{F} = \{F_1,\ F_2, ...,\ F_n\}$ is a basis of $\mathbb{R}^n$.

[10] **2**. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{bmatrix}$
(a) Find a basis of *null* A.
(b) Find a basis of *im* A.

[10] **3**. Consider the basis $\mathcal{F} = \{F_1, F_2\}$ of $\mathbb{R}^2$ where $F_1 = [1, -1]^T$, and $F_2 = [-1, 2]^T$.Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator on $\mathbb{R}^2$ with $M_{\mathcal{F}}[T] = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$.
(a) Find $C_{\mathcal{F}}(E_1)$ and $C_{\mathcal{F}}(E_2)$ where $\mathcal{E} = \{E_1, E_2\}$ is the standard basis of $\mathbb{R}^2$.
(b) Find $C_{\mathcal{F}}(T(E_1))$ and $C_{\mathcal{F}}(T(E_2))$.
(c) Find $T(E_1)$, $T(E_2)$ and the standard matrix of T.
(d) Is T invertible? Give reasons.

[10] **4**. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. We define the *kernel* of T (denoted by $\ker T$) and the image of T (denoted by imT) by
$\ker T = \{X \in \mathbb{R}^n \mid T(X) = 0\}$ and $imT = \{T(X) \mid X \in \mathbb{R}^n\}$.
(a) Prove that $\ker T$ is a subspace of $\mathbb{R}^n$.
(b) Prove that imT is a subspace of $\mathbb{R}^m$.
(c) Is it true that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, $\dim(\ker T) + \dim(imT) = n$? No explanation needed. **Your answer:**
(d) Is it true that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, $\ker T \cap imT = \emptyset$? No explanation needed. **Your answer:**

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QUIZ 1
Tuesday, January 25, 2005 **Duration: 30 minutes.**

[7] **1**. Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$.
(a) Find $nullA$.
(b) Find imA.
(c) Are the columns of A independent? Prove your answer.

[8] **2**. In this question, X, Y, Z are vectors in $\mathbb{R}^n$. Prove or disprove each of the following statements **using only the definitions** of subspaces $\mathbb{R}^n$, of linearly independent (dependent):
(a) If U is a subspace of $\mathbb{R}^n$, and both $2Y$ and $X - Y$ are in U then X and Y are in U.
(b) For any vectors X, Y in R^n, $\{X,\ Y,\ 2X - Y\}$ is linearly dependent.
(c) If $\{X, Y, Z\}$ is is linearly dependent then $\{X, Y\}$ is is linearly dependent.

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QUIZ 2
Thursday, February 10, 2005 **Duration: 30 minutes.**

[4] **1**. Given that A is a 4×3 matrix and $rankA = 2$. Can A have independent rows? Explain.

[4] **2**. Prove the following statement:
If $\{X, Y\}$ is a basis of $\mathbb{R}^2$ then $\{X - Y, X + Y\}$ is also a basis of $\mathbb{R}^2$.

[7] **2**.Let $A = \begin{bmatrix} 2 & -1 & 1 & 3 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.
(a) Are the columns of A linearly independent? Explain.
(b) Solve the system $AX = 0$.
(c) Find a basis of $null\ A$.
(d) Find a basis of $im\ A$.

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QUIZ 3
Thursday, March 3, 2005 **Duration: 30 minutes.**

[7] **1**. Consider the basis $\mathcal{F} = \{F_1, F_2\}$ of $\mathbb{R}^2$ where $F_1 = [2,\ 3]^T$ and $F_2 = [3,\ 4]^T$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator so that the $\mathcal{F}$-matrix of T is $M_{\mathcal{F}}(T) = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$.
(a) Find $T(F_1)$ and $T(F_2)$.
(b) Find the standard matrix of T.

[8] **2**.Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The **kernel** of T (denoted by $\ker T$) is defined by $\ker T = \{X \in \mathbb{R}^n \mid T(X) = 0\}$.
(a) Prove that $\ker T$ is a subspace of $\mathbb{R}^n$.
(b) Prove that if $S \circ T = 1_{\mathbb{R}^n}$ for some linear transformation $S : \mathbb{R}^m \to \mathbb{R}^n$ then $\ker T = \{0\}$.

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QUIZ 4

Thursday, March 24, 2005 **Duration: 30 minutes.**

[4] **1**. Let $T : \mathbb{M}_{2\times 2} \to \mathbb{R}$ be the function defined by $T(A) = rank A$ for all $A \in \mathbb{M}_{2\times 2}$. Is T a linear transformation? Explain.

[4] **2**. Let $U = span\{1 + x,\ 2 - x^2,\ x^2 + 2x\}$ be a subspace of $\mathbb{P}_2$? Find a basis of U.

[7] **3**. Let $\overrightarrow{v}_1,\ \overrightarrow{v}_2,\ ...,\ \overrightarrow{v}_n$ be vectors in a vector space V and define $T : \mathbb{R}^n \to V$ by $T\left([x_1,\ x_2, ...,\ x_n]^T\right) = x_1\overrightarrow{v}_1 + x_2\overrightarrow{v}_2 + ... + x_n\overrightarrow{v}_n$. Prove the following statements:

(a) If T is one-to-one then $\{\overrightarrow{v}_1,\ \overrightarrow{v}_2,\ ...,\ \overrightarrow{v}_n\}$ is linearly independent.

(b) If $\{\overrightarrow{v}_1,\ \overrightarrow{v}_2,\ ...,\ \overrightarrow{v}_n\}$ is linearly independent then T is one-to-one.

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QUIZ 5

Thursday, April 7, 2005 **Duration: 30 minutes.**

[7] **1**. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation defined by

$$T\left(a + bx + cx^2\right) = (a + b) + (b + c)\,x + (c + a)\,x^2.$$

Find $M_{\mathcal{DB}}(T)$ where $\mathcal{B} = \{1,\ x,\ x^2\}$ and $\mathcal{D} = \{1 + x,\ x + x^2,\ x^2 + 1\}$.

[8] **2**. Let $X, Y \in \mathbb{R}^n$. Prove the following statements:

(a) If $\|X\| = \|Y\|$ then $X + Y$ and $X - Y$ are orthogonal.

(b) If $\|X + Y\|^2 = \|X\|^2 + \|Y\|^2$ then X and Y are orthogonal.