UNIVERSITY OF CALGARY

Faculty of Science
Department of Mathematics & Statistics

MIDTERM - MATH 311 - L01

February 25, 2009

Solutions

Your family name: ______________________

Your first name: ______________________

Your signature: ______________________

Your student number: ______________________

INSTRUCTIONS:

I. Fill out the above information BEFORE starting this exam.
II. **Show all your work**, use the back of the previous page for rough work and clearly insert the main steps and answers in the provided space.
III. Calculators are not allowed, and no other material.
IV. There are 4 questions and 4 pages to this exam.
V. Time allowed is 50 minutes.

PROBLEM	#1	#2	#3	#4	TOTAL
MARKS	/6	/3	/4	/7	/20

Question 1 (6 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 2 & 4 & -1 \end{bmatrix}.$$

[2] a) With justification, find a basis for $row(A)$ and find $dim(row(A))$.

$$\begin{matrix} -R_1+R_2 \\ -2R_1+R_3 \end{matrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} -R_2+R_3 \\ R_2+R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the leading 1's are in Rows 1 and 2, so these rows are independent.

Basis: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ Dimension: 2

[2] b) With justification, find a basis for $null(A)$ and find $dim(null(A))$.

the general solution is

$$\begin{aligned} x &= -2s \\ y &= s \\ z &= 0 \end{aligned} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$ Dimension: 1.

[2] c) With justification, find a basis for $col(A)$ and find $dim(col(A))$.

the leading 1's are in columns 1 and 3 so the corresponding columns of A form a basis

Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \right\}$ Dimension: 2

Question 2 (3 points)
Let A be an $n \times n$ matrix and λ an eigenvalue of A. Show that

$$E_\lambda(A) = \{X \text{ in } \mathbb{R}^n : AX = \lambda X\}$$

is a subspace of $\mathbb{R}^n$.

$E_\lambda(A) = \text{null}(A - \lambda I)$ which we know is a subspace.

Or directly:

(1) $A0 = 0 = \lambda 0$ so $0 \in E_\lambda(A)$

(2) ~~If~~ $X_1, X_2 \in E_\lambda(A)$ then $AX_1 = \lambda X_1$ and $AX_2 = \lambda X_2$

so $A(X_1 + X_2) = AX_1 + AX_2$

$= \lambda X_1 + \lambda X_2$

$= \lambda(X_1 + X_2)$ so $X_1 + X_2 \in E_\lambda(A)$

(3) If $X \in E_\lambda(A), a \in \mathbb{R}$

then $AX = \lambda X$

so $A(aX) = a(AX) = a(\lambda X) = \lambda(aX)$, so $aX \in E_\lambda(A)$.

Question 3 (4 points)
Show with full justification that an orthogonal set in $\mathbb{R}^n$ is linearly independent.

Suppose $\{X_1, \dots, X_k\}$ is orthogonal

i.e. $X_i \cdot X_j = 0$ for $i \neq j$ and $X_i \neq 0$

Now if $a_1 X_1 + \dots + a_k X_k = 0$

then $0 = X_i \cdot 0 = X_i \cdot (a_1 X_1 + \dots + a_k X_k)$

$= a_1 X_i \cdot X_1 + \dots + a_k X_i \cdot X_k$

$= a_i X_i \cdot X_i$

But $X_i \cdot X_i \neq 0$ so $a_i = 0$

Thus $a_1 = a_2 = \dots = a_k = 0$

and $\{X_1, \dots X_k\}$ is independent.

Question 4 (7 points) Consider $\mathbb{P}_3$ the set of polynomials of degree at most 3.
[2] a) With justification, give a basis of $\mathbb{P}_3$ and find its dimension.

Since $S = \{1, x, x^2, x^3\}$ are polynomials of different degree, then they are independent.
Also $a + bx + cx^2 + dx^3 = a \cdot 1 + b \cdot x + c \cdot x^2 + d \cdot x^3$
so S is also a spanning set.

Basis: $\{1, x, x^2, x^3\}$ Dimension: 4

[3] b) With justification, is $S = \{x^2, x^2 + x, x^2 + x + 1\}$ an independent set in $\mathbb{P}_3$?

Suppose $a(x^2) + b(x^2 + x) + c(x^2 + x + 1) = 0$

then $(a + b + c)x^2 + (b + c)x + c \cdot 1 = 0$

$\therefore$ $a + b + c = 0$, $b + c = 0$, $c = 0$ $\therefore$ $a = b = c = 0$

So it is independent.

[2] b) With justification, extend S to a basis of $\mathbb{P}_3$.

$x^3 \in \mathbb{P}_3$, but $x^3 \notin \operatorname{span} S$ since $S \subseteq \mathbb{P}_2$

so $S \cup \{x^3\}$ is linearly independent

and because $\dim \mathbb{P}_3 = 4$, it must be a basis.

Basis: $S \cup \{x^3\}$.