UNIVERSITY OF CALGARY

Faculty of Science

Department of Mathematics & Statistics

Quiz #~~4~~2 - MATH 311 - L01 - T01/T02

Tuesday March 17, 2009

(Solutions)

Your family name: ______________________

Your first name: ______________________

Your signature: ______________________

Your student number: ______________________

INSTRUCTIONS:

I. Fill out the above information BEFORE starting.

II. **Show all your work**, use the back of the previous page for rough work and clearly insert your solution in the space provided space.

III. Calculators are not allowed, and no other material.

IV. There are 2 questions and 3 pages to this exam.

V. Time allowed is 50 minutes.

MARKS:	#1: /4	#2: /6	Total: /10

Question 1 (4 points) Consider the following linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ given by

$$T(a + bx + cx^2) = a + (a - b)x + (a - b - c)x^2$$

[2] **a)** With justification, find $ker(T)$.

$a+bx+cx^2 \in \ker(T) \iff T(a+bx+cx^2)=0$

$\iff a+(a-b)x+(a-b-c)x^2=0$

$\iff \begin{cases} a=0 \\ a-b=0 \\ a-b-c=0 \end{cases}$

$\iff a=b=c=0$

Answer: $ker(T) =$ $\{0\}$

[2] **b)** With justification, find $im(T)$, and argue whether ~~T is an isomorphism.~~

$$\boxed{\begin{matrix} \dim \mathbb{P}_2 = \dim \ker(T) + \dim \operatorname{im} T \\ 3 \qquad\quad 0 \quad + \_\_\_ \end{matrix}}$$

so $\dim(\operatorname{im} T) = 3$

Since $\operatorname{im} T \subseteq \mathbb{P}_2$ is a $\dim(\mathbb{P}_2) = 3$

$\therefore \operatorname{im} T = \mathbb{P}_2$

and T is onto

T is 1-1 since $\ker T = \{0\}$

T is onto since $\operatorname{im} T = \mathbb{P}_2$

Answer: $im(T) =$ $\mathbb{P}_2$ Isomorphism: yes

Question 2 (6 points) Consider the following linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ given by

$$T(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T(x-1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \text{and } T(x^2 - x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since $B = \{1, x-1, x^2 - x\}$ is a basis of $\mathbb{P}_2$, the above does uniquely define a linear transformation.

[2] **a)** Compute $T(1)$, $T(x)$ and $T(x^2)$.

$x = (x-1) + 1$ so $T(x) = T(x-1) + T(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x^2 = (x^2 - x) + (x-1) + 1$

so $T(x^2) = T(x^2 - x) + T(x-1) + T(1)$

$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Answer: $T(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $T(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $T(x^2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

[2] **b)** With justification, is T onto?

Yes $\mathbb{R}^2 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \subseteq \text{im} T \subseteq \mathbb{R}^2$

so $\mathbb{R}^2 = \text{im} T$.

[2] **c)** With justification, find $ker(T)$ and $dim(ker(T))$.

$\underbrace{\dim \mathbb{P}_2}_{3} = \underbrace{\dim \ker T}_{\therefore 1} + \underbrace{\dim(\text{im} T)}_{2}$

so $\dim(\ker T) = 1$

but $T(x^2 - 2x - 1) = 0$ from above.

so $\{x^2 - 2x - 1\}$ is a basis for $\ker(T)$.

Answer: $ker(T) =$ $\text{span}\{x^2 - 2x - 1\}$ $dim(ker(T)) =$ 1

or any multiple.

UNIVERSITY OF CALGARY

Faculty of Science
Department of Mathematics & Statistics

Quiz #~~1~~2 - MATH 311 - L01 - T03

Thursday March 19, 2009

SOLUTIONS

Your family name: ______________________________

Your first name: ______________________________

Your signature: ______________________________

Your student number: ______________________________

INSTRUCTIONS:
I. Fill out the above information BEFORE starting.
II. **Show all your work**, use the back of the previous page for rough work and clearly insert your solution in the space provided space.
III. Calculators are not allowed, and no other material.
IV. There are 2 questions and 3 pages to this exam.
V. Time allowed is 50 minutes.

MARKS:	#1: /4	#2: /6	Total: /10

Question 1 (4 points) Consider the following linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ given by

$$T(a + bx + cx^2) = \begin{bmatrix} a - b \\ a - c \end{bmatrix}.$$

[2] a) With justification, find a basis for $ker(T)$ and $dim(ker(T))$.

$$a+bx+cx^2 \in \ker(T) \iff T(a+bx+cx^2) = 0$$

$$\iff \begin{bmatrix} a-b \\ a-c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff a-b=0 \text{ and } a-c=0$$

$$\iff a=b=c$$

So $\ker(T) = \{a + ax + ax^2 : a \in \mathbb{R}\}$
$= \text{span}\{1+x+x^2\}$

Answer: Basis $\{1+x+x^2\}$ Dimension 1

[2] b) With justification, find a basis for $im(A)$ and $dim(im(T))$.

$$\underset{\underset{3}{\parallel}}{\dim \mathbb{P}_2} = \underset{\underset{1}{\parallel}}{\dim \ker T} + \dim(\operatorname{im} T)$$

So $\dim(\operatorname{im} T) = 2$

but $\operatorname{im} T \subseteq \mathbb{R}^2$ and $\dim \mathbb{R}^2 = 2$

so $\operatorname{im} T = \mathbb{R}^2$.

Answer: Basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ Dimension 2

Question 2 (6 points) Consider the following linear transformation $T : \mathbb{P}_1 \to \mathbb{R}^2$ given by

$$T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}.$$

[1] a) Compute $T(1)$ and $T(x)$.

Answer: $T(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $T(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

[2] b) With justification, show that T an isomorphism.

T sends the basis $\{1, x\}$ of $\mathbb{P}_1$ to the basis $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ of $\mathbb{R}^2$.

(One can also show that $\ker(T) = 0$ (so T is 1-1) and T is onto)

[1] c) Since T is an isomorphism, there must be a linear transformation $S : \mathbb{R}^2 \to \mathbb{P}_1$ such that $ST = 1_{\mathbb{P}_1}$ and $TS = 1_{\mathbb{R}^2}$. Find $S(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$ and $S(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$.

Since $ST = 1_{\mathbb{P}_1}$ then $1 = ST(1) = S\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x = ST(x) = S\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Answer $S(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = 1$ $S(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = x$

[2] c) With justification, find $S(\begin{bmatrix} a \\ b \end{bmatrix})$ for any $a, b \in \mathbb{R}$.

$$\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b-a)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so $$S\begin{bmatrix} a \\ b \end{bmatrix} = a\,S\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b-a)\,S\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= a \cdot 1 + (b-a)\,x.$$