UNIVERSITY OF CALGARY

Faculty of Science
Department of Mathematics & Statistics

Quiz #3 - MATH 311 - L01 - T01 +T02
Tuesday April 7, 2009

SOLUTIONS

Your family name: ____________________

Your first name: ____________________

Your signature: ____________________

Your student number: ____________________

INSTRUCTIONS:
I. Fill out the above information BEFORE starting.
II. **Show all your work**, use the back of the previous page for rough work and clearly insert your solution in the space provided space.
III. Calculators are not allowed, and no other material.
IV. There are 2 questions and 3 pages to this exam.
V. Time allowed is 50 minutes.

MARKS:	#1: /5	#2: /5	Total: /10

Question 1 (5 points) Consider the following linear operator $T : \mathbb{P}_2 \to \mathbb{P}_2$ given by

$$T(a + bx + cx^2) = (a - b) + (b - c)x + (c - a)x^2$$

[3] **a)** Compute $M_B(T)$, the matrix of T corresponding to the standard basis $B = \{1, x, x^2\}$ of $\mathbb{P}_2$.

$$M_B(T) = \left[C_B T(1) \quad C_B T(x) \quad C_B T(x^2) \right] = \left[C_B(1-x^2) \quad C_B(1-x) \quad C_B(-x+x^2) \right.$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Answer: $M_B(T) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

[2] **b)** With justification, use the matrix $M_B(T)$ from part a) to compute $dim(ker(T))$ and $dim(im(T))$.

$$M_B(T) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$\begin{array}{c} R_1 + R_3 \\ -R_2 \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$-R_2 + R_3 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

so $\dim(\operatorname{im} T) = \operatorname{rank} T = \operatorname{rank} M_B(T) = 3$.

$\underset{3}{\dim \mathbb{P}_2} = \underset{\therefore 0}{\dim \ker T} + \underset{3}{\dim \operatorname{im} T}$

(or $\dim \ker T = \dim(\operatorname{null} M_B(T)) = 0$).

Answer: $dim(ker(T)) =$ 0 $dim(im(T)) =$ 3

Question 2 (5 points)

[2] a) Clearly define what it means for an $n \times n$ matrix P to be orthogonal.

It means that the columns (or rows) $\{X_1, \ldots, X_n\}$ form an orthonormal set, i.e.

$X_i \cdot X_j = 0$ for $i \neq j$

$\|X_i\| = 1$

[1] b) Clearly define what it means to orthogonally diagonalize an $n \times n$ matrix A.

It means that there is an orthogonal matrix P

$P^T A P = D$ is diagonal.

[2] c) With justification, show that if A is orthogonally diagonalizable, then A is symmetric.

Suppose $P^T A P = D$ for some orthogonal P

$\therefore$ $A = PDP^T$ since $P^{-1} = P^T$

$\therefore$ $A^T = (PDP^T)^T$

$= P^{T^T} D^T P^T$

$= PDP^T$ (since $D^T = D$)

$= A$.

So A is symmetric //

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Quiz #3 - MATH 311 - L01 - T03

Tuesday April 9, 2009

SOLUTION

Your family name: ____________________

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Your signature: ____________________

Your student number: ____________________

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MARKS:	#1: /5	#2: /5	Total: /10

Question 1 (5 points) Consider the following linear operator $T : \mathbb{P}_2 \to \mathbb{P}_2$ given by

$$T(a + bx + cx^2) = a + (b - a)x + (c - a - b)x^2$$

[3] **a)** Compute $M_B(T)$, the matrix of T corresponding to the standard basis $B = \{1, x, x^2\}$ of $\mathbb{P}_2$.

$$M_B(T) = \left[C_B T(1) \quad C_B T(x) \quad C_B T(x^2) \right]$$

$$= \left[C_B(1 - x - x^2) \quad C_B(x - x^2) \quad C_B(x^2) \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Answer: $M_B(T) =$ ________________

[2] **b)** With justification, use the matrix $M_B(T)$ from part a) to compute the inverse $T^{-1}(a + bx + cx^2)$.

$$M_B(T^{-1}) = [M_B(T)]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

So

$$T^{-1}(a + bx + cx^2) = C_B^{-1} M_B(T^{-1}) C_B(a + bx + cx^2)$$

$$= C_B^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = C_B^{-1} \begin{bmatrix} a \\ a+b \\ 2a+b+c \end{bmatrix}$$

$$= a + (a+b)x + (2a+b+c)x^2$$

Answer: $T^{-1}(a + bx + cx^2) =$ ________________

Question 2 (5 points) Let U be a subspace of $\mathbb{R}^n$.

[1] a) Clearly define $U^\perp$.

$\mathcal{U}$

$U^\perp = \{ X \in \mathbb{R}^n : X \cdot Y = 0 \text{ for all } Y \in \cancel{\mathbb{R}^n} \}$

[2] b) Prove that $U^\perp$ is a subspace of $\mathbb{R}^n$.

① $0 \in \mathcal{U}^\perp$ since $0 \cdot Y = 0$ for all $Y \in \mathcal{U}$

② If $X_1, X_2 \in \mathcal{U}^\perp$, then $X_1 \cdot Y = 0$ and $X_2 \cdot Y = 0$ for all $Y \in \mathcal{U}$

so $(X_1 + X_2) \cdot Y = X_1 \cdot Y + X_2 \cdot Y = 0 + 0 = 0$ for all $Y \in \mathcal{U}$.

③ If $X \in \mathcal{U}^\perp$ and $a \in \mathbb{R}$ then $(aX) \cdot Y = a(X \cdot Y) = a0 = 0$ for all $Y \in \mathcal{U}$

so $aX \in \mathcal{U}^\perp$.

[2] c) With justification, show that $U \cap U^\perp = \{0\}$.

Let $X \in \mathcal{U} \cap \mathcal{U}^\perp$

then $X \cdot X = 0$ since $X \in \mathcal{U}$ and $X \in \mathcal{U}^\perp$

ie $\|X\|^2 = 0$

so $X = 0$.