ASSIGNMENT 4.

MATH 311 **FALL 2010**

1. *Let $T : V \to W$ be a linear transformation, and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be vectors in V.*

   (1) *If $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ is independent in W, show that $\mathcal{B}$ is independent in V.* 4 marks

   (2) *If T is onto and $\mathcal{B}$ spans V, show that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ spans W.* 4 marks

2. *Given $a \in \mathbb{R}$, define the evaluation transformation $E_a : \mathbb{P}_n \to \mathbb{R}$ by $E_a[p(x)] = p(a)$ for all polynomials $p(x)$ in $\mathbb{P}_n$. You may assume that E_a is linear.*

   (1) *Show that E_a satisfies $E_a(x^k) = [E_a(x)]^k$ for each $k = 0, 1, 2, \ldots$. [Note: $x^0 = 1$.]* 4 marks

   (2) *If $T : \mathbb{P}_n \to \mathbb{R}$ is any linear transformation satisfying $T(x^k) = [T(x)]^k$ for each $k = 0, 1, 2, \ldots$, show that $T = E_a$ for some $a \in \mathbb{R}$.* [Hint: Given such a T, take $a = T(x)$.] 4 marks

3. *Define $T : \mathbb{P}_n \to \mathbb{R}$ by taking $T[p(x)]$ to be the sum of all the coefficients of $p(x)$.*

   (1) *Use the dimension theorem to show that $dim[ker(T)] = n$.* 4 marks

   (2) *Conclude that $\mathcal{B} = \{x - 1,\, x^2 - 1, \cdots, x^n - 1\}$ is a basis of $ker(T)$.* 4 marks

4. *Let U be an invertible $n \times n$ matrix, and define.* $T : \mathbb{M}_{nn} \to \mathbb{M}_{nn}$ by $T(A) = UA$ for every $A \in \mathbb{M}_{nn}$.

   (1) *Show that T is an isomorphism.* 4 marks

   (2) *Does every isomorphism $S : \mathbb{M}_{22} \to \mathbb{M}_{22}$ arise as in part (1)? Defend your answer.*

   [Hint: *transpose.*] 4 marks

5. *Let $T : V \to W$ be a linear transformation.*

   (1) *If T is $1:1$ and $TR = TR_1$ for transformations $R, R_1 : U \to V$, show that $R = R_1$.* 4 marks

   (2) *If T is onto and $ST = S_1T$ for transformations $S, S_1 : W \to U$, show that $S = S_1$.* 4 marks

Total: 40 marks