SOLUTIONS to ASSIGNMENT 1.

MATH 311 **FALL 2010**

ASSIGNMENT 1

1. *If $\mathbb{R}^n = span\{X_1, X_2, \cdots, X_k\}$ and $A \neq 0$ is $m \times n$, show that $AX_i \neq 0$ for some i.* 7 marks

   Solution: Suppose $AX_i = 0$ for each i. If $X \neq 0$ is in $\mathbb{R}^n$, write $X = t_1X_1 + \cdots + t_kX_k$, t_i in $\mathbb{R}$. Then $AX = t_1AX_1 + \cdots + t_kAX_k = t_10 + \cdots + t_k0 = 0$. In particular, $AE_j = 0$ for every column E_j of I_n. But then $A = AI = A[E_1 \;\; E_2 \;\; \cdots \;\; E_n] = [AE_1 \;\; AE_2 \;\; \cdots \;\; AE_n] = 0$, contrary to assumption.

2. *Let A denote an $m \times n$ matrix. Show that $dim[null(A)] = dim[null(AV)]$ for any invertible $n \times n$ matrix V. [Hint: If $AX = 0$ then $AV(V^{-1}X) = 0$.]* 7 marks

   Solution: Recall that $null(A) = \{X \text{ in } \mathbb{R}^n \mid AX = 0\}$. Let $\{X_1, X_2, \cdots, X_k\}$ be a basis of $null(A)$, so that $dim[null(A)] = k$. It suffices to show that $B = \{V^{-1}X_1, V^{-1}X_2, \cdots, V^{-1}X_k\}$ is a basis of $null(AV)$. Note that each of these vectors is actually in $null(AV)$ because $(AV)(V^{-1}X_i) = AX_i = 0$ for each i.

   Independence: Let $t_1(V^{-1}X_1) + t_2(V^{-1}X_2) + \cdots + t_k(V^{-1}X_k) = 0$, t_i in $\mathbb{R}$. Taking V^{-1} out as a common factor gives $V^{-1}(t_1X_1 + t_2X_2 + \cdots + t_kX_k) = 0$. Hence $t_1X_1 + t_2X_2 + \cdots + t_kX_k = 0$, so each $t_i = 0$ by the independence of $\{X_1, X_2, \cdots, X_k\}$

   Spanning: Let Y be a vector in $null(AV)$; we must show that it is a linear combination of the $V^{-1}X_i$. We have $(AV)Y = 0$ so VY is in $null(A)$. Hence $VY = r_1X_1 + r_2X_2 + \cdots + r_kX_k$ for some r_i in $\mathbb{R}$ because the X_i span $null(A)$. But then $Y = r_1(V^{-1}X_1) + r_2(V^{-1}X_2) + \cdots + r_k(V^{-1}X_k)$, as required.

3. *Show that $\{x^2 + 2,\, 3x - x^3,\, 4 + x + x^3, x\}$ spans $\mathbf{P}_3$.* 6 marks

   Solution: Write $U = span\{x^2 + 2, 3x - x^3, 1 + x + x^3, x\}$ and use Theorem 2 §6.2 repeatedly. First $x \in U$; then $x^3 \in U$ because $3x - x^3 \in U$; then $1 \in U$ because $4 + x + x^3 \in U$; finally $x^2 \in U$ because $x^2 + 2 \in U$. This shows that $\{1, x, x^2, x^3\} \subseteq U$ so (since U is a subspace), $span\{1, x, x^2, x^3\} \subseteq U$. But $span\{1, x, x^2, x^3\} = \mathbf{P}_3$ so we have shown that $\mathbf{P}_3 \subseteq U$. Since clearly $U \subseteq \mathbf{P}_3$, we have $U = \mathbf{P}_3$.

4. *Let $U = span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_k\}$ in a vector space V. If $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_k\mathbf{v}_k$ where $a_2 \neq 0$, show that $U = span\{\mathbf{v}_1, \mathbf{w}, \mathbf{v}_3, \cdots, \mathbf{v}_k\}$.* 7 marks

   Solution: We first show that $\mathbf{v}_2$ is in $span\{\mathbf{v}_1, \mathbf{w}, \mathbf{v}_3, \cdots, \mathbf{v}_k\}$. In fact, since $a_2 \neq 0$ we have $\mathbf{v}_2 = -\frac{a_1}{a_2}\mathbf{v}_1 + \frac{1}{a_2}\mathbf{w} - \frac{a_3}{a_2}\mathbf{v}_3 - \cdots - \frac{a_k}{a_2}\mathbf{v}_k$. With this we have $U \subseteq span\{\mathbf{v}_1, \mathbf{w}, \mathbf{v}_3, \cdots, \mathbf{v}_k\}$ by Theorem 2 §6.2. On the other hand, since $\mathbf{w}$ is in U we have $\{\mathbf{v}_1, \mathbf{w}, \mathbf{v}_3, \cdots, \mathbf{v}_k\} \subseteq U$. Hence Theorem 2 §6.2 (again) shows that $span\{\mathbf{v}_1, \mathbf{w}, \mathbf{v}_3, \cdots, \mathbf{v}_k\} \subseteq U$, so we have $span\{\mathbf{v}_1, \mathbf{w}, \mathbf{v}_3, \cdots, \mathbf{v}_k\} = U$ as required.

5. *Among all independent sets in a vector space V, let $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ be an independent set where k is as large as possible. Show that B is a basis of V.* 7 marks

   Solution: Write $U = span\ B$. Since B is independent, it remains to show that $U = V$. Assume not; we look for a contradiction. Since $U \subseteq V$ and $U \neq V$, choose a vector $\mathbf{w} \in V$ such that $\mathbf{w} \notin U$. Since B is independent, Lemma 1 §6.4 shows that $\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is also independent. But $\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ clearly contains more vectors than B, contradicting the maximality of B.

6. *Find the dimension of the subspace U of* $\mathbb{R}^5$ if $U = \left\{ \begin{bmatrix} a-b \\ c+2b \\ 3b+c \\ c-a \\ 3a-2b+5c \end{bmatrix} \mid a, b, c \text{ in } \mathbb{R} \right\}$. 6 marks

Solution: Each vector X in U has the form $X = aX_1 + bX_2 + cX_3$ where $X_1 = [1 \ \ 0 \ \ 0 \ \ -1 \ \ 3]^T$, $X_2 = [-1 \ \ 2 \ \ 3 \ \ 0 \ \ -2]^T$, and $X_3 = [0 \ \ 1 \ \ 1 \ \ 1 \ \ 5]^T$. Hence $U = span\{X_1, X_2, X_3\}$, so $dim(U) = 3$ provided we can show that $\{X_1, X_2, X_3\}$ is independent. But $aX_1 + bX_2 + cX_3 = 0$ implies (see the vector in the definition of U) that $a - b = 0$, $c + 2b = 0$, $3b + c = 0$, $c - a = 0$, and $3a - 2b + 5c = 0$. These equations imply that $a = b = c = 0$, as required.

Total: 40 marks