SOLUTIONS to ASSIGNMENT 2.

MATH 311 **FALL 2010**

1. (a) *Show that* $\mathbb{P}_3 = span\{x^2+2,\ x+1,\ x^3-2x,\ 5x\}$. 2 marks

   (b) *Suppose A and B are nonzero* $n \times n$ *matrices. If* $A^T = A$ *and* $B^T = -B$*, show that* $\{A, B\}$ *is an independent subset of* $\mathbb{M}_{nn}$. 3 marks

   (c) *If* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *is an independent set in a vector space V, show that* $\{\mathbf{u}+\mathbf{v},\ \mathbf{v}+\mathbf{w},\ \mathbf{w}+\mathbf{u}\}$ *is also independent.* 3 marks

   Solution. (a). Write $S = span\{x^2+2,\ x+1,\ x^3-2x,\ 5x\}$. Then $x \in S$ because $5x \in S$; then $1 \in S$ because $x+1 \in S$; then $x^2 \in S$ because $x^2+2 \in S$; and finally $x^3 \in S$ because $x^3 - 2x \in S$. By Theorem 2 §6.2 it follows that $\mathbb{P}_3 = span\{1, x, x^2, x^3\} \subseteq S$. Since clearly $S \subseteq \mathbb{P}_3$, we have $S = \mathbb{P}_3$.

   (b). Let $sA + tB = 0$, $s, t \in \mathbb{R}$; we must show that $s = 0 = t$. Transposing gives $sA - tB = 0$, and adding these gives $2sA = 0$. As $A \neq 0$ we obtain $s = 0$. Hence $tB = 0$ so $t = 0$ because $B \neq 0$.

   (c). Suppose $r(\mathbf{u}+\mathbf{v}) + s(\mathbf{v}+\mathbf{w}) + t(\mathbf{w}+\mathbf{u}) = \mathbf{0}$, that is $(r+t)\mathbf{u} + (r+s)\mathbf{v} + (s+t)\mathbf{w} = \mathbf{0}$. Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent, this means $r+t = 0$, $r+s = 0$, $s+t = 0$. The only solution is $r = s = t = 0$.

2. *Let U and W be subspaces of a vector space V, and assume that* $dim(U) = 2$*. Show that either* $U \subseteq W$ *or* $dim(U \cap W) \leq 1$. 8 marks

   Solution. We have $(0\} \subseteq U \cap W \subseteq U$ so $dim(U \cap W) = 0, 1, 2$ by Theorem 2 §6.4. Hence either $dim(U \cap W) \leq 1$ or $dim(U \cap W) = 2$. But in this last case we have $U \cap W \subseteq U$ and $dim(U \cap W) = dim(U)$, so $U \cap W = U$ by Theorem 2 §6.4. It follows that $U \subseteq W$.

3. *Let A be an* $m \times n$ *matrix.*

   (a) *If* $rank(A) = n$ *and* $AB = 0$ *where B is* $n \times k$*, show that* $B = 0$. 4 marks

   (b) *If* $rank(A) = m$ *and* $CA = 0$ *where C is* $k \times m$*, show that* $C = 0$. 4 marks

   Solution. (a) Theorem 3 §5.4 shows that $AX = 0$, $X \in \mathbb{R}^n$, implies $X = 0$. Write $B = [B_1\ \ B_2\ \ \cdots\ \ B_k]$ where B_j denotes column j of B. Then $AB = [AB_1\ \ AB_2\ \ \cdots\ \ AB_k]$, so $AB = 0$ gives $AB_j = 0$ for all j, that is $B_j = 0$ for all j, that is $B = 0$.

   (b). If $CA = 0$ then $A^TC^T = (CA)^T = 0$. Since A^T is $n \times m$, we are done by (1) if $rank(A^T) = m$ (then $C^T = 0$ by (1), and so $C = 0$). But $rank(A^T) = rank(A) = m$ by Corollary 1, Theorem 1 §5.4.

4. *Let A be an* $m \times n$ *matrix, with columns* $C_1, C_2, \cdots, C_n$*. If* $\{C_1, C_2, \cdots, C_n\}$ *is an orthogonal set in* $\mathbb{R}^m$*, show that* A^TA *is a diagonal* $n \times n$ *matrix, and describe its diagonal entries.* 8 marks

   Solution. By the definition of matrix multiplication, the (i, j)-entry of A^TA is $C_i^TC_j = C_i \bullet C_j$. Since $\{C_1, C_2, \cdots, C_n\}$ is orthogonal, this shows that the (i, j)-entry of A^TA is zero if $i \neq j$, while, if $j = i$ it is $C_i \bullet C_i = \|C_i\|^2$. Hence $A^TA = diag(\|C_1\|^2, \|C_2\|^2, \cdots, \|C_n\|^2)$.

5. *If A is an* $m \times n$ *matrix, show that* $\lambda \geq 0$ *for every eigenvalue* λ *of* A^TA. [Hint: $\|X\|^2 = X^TX$ for every column X in $\mathbb{R}^n$.] 8 marks

   Solution. Let X be an eigenvector of A^TA corresponding to λ. Then $X \neq 0$ and $(A^TA)X = \lambda X$. Left multiply by X^T to obtain $X^T(A^TA)X = X^T(\lambda X)$, that is $(AX)^T(AX) = \lambda(X^TX)$. Hence the Hint gives $\|AX\|^2 = \lambda\|X\|^2$, so, as $\|X\| \neq 0$, we have $\lambda = \frac{\|AX\|^2}{\|X\|^2} \geq 0$.

Total: 40 marks