SOLUTIONS to ASSIGNMENT 3.

MATH 311 **FALL 2010**

1. *Let* $A = \begin{bmatrix} 1 & -1 & -1 & 2 & 2 & 3 \\ -2 & 2 & 3 & -6 & -3 & -11 \\ -1 & 1 & 2 & -4 & -3 & -8 \\ 0 & 0 & 1 & -2 & 1 & -5 \end{bmatrix}$.

$$\begin{matrix} 1 & -1 & -1 & 2 & 2 & 3 \\ -2 & 2 & 3 & -6 & -3 & -11 \\ -1 & 1 & 2 & -4 & -3 & -8 \\ 0 & 0 & 1 & -2 & 1 & -5 \end{matrix}\text{, row echelon form: } \begin{matrix} 1 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

(1) Find bases of $row(A)$ and of $col(A)$, and so determine their dimensions. 4 marks

(2) Find bases of $null(A)$ and of $im(A)$, and so determine their dimensions. 4 marks

Solution. (1). Standard Gaussian elimination gives $A \to R$ where $R = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is reduced row-echelon. So, by the Rank Theorem (Theorem 1 §5.4), the first three rows of R are a basis of $row(A)$, whence $dim[row(A)] = 2 = rank(A)$. Moreover, the leading 1s are in columns 1, 3 and 5 of R, so columns 1, 3 and 5 of A are a basis of $col(A)$, again by the Rank Theorem.

(2). The reduced row echelon form of A is $\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ so the leading variables are x_1, x_3 and x_5. Assigning the non-leading variables as parameters: $x_2 = r$, $x_4 = s$ and $x_6 = t$, we solve for the leading variables: $x_1 = r + 2t$, $x_3 = 2s + 5t$, and $x_5 = 0$. Thus the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = rX_1 + sX_2 + tX_3 \text{ where } X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ X_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then $\{X_1, X_2, X_3\}$ is a basis of $null(A)$ by Theorem 2 §5.4, so $dim[null(A)] = 3$. Note that A is 4×6 here with rank $r = dim[col(A)] = 3$ so $dim[null(A)] = 3 = 6 - 3 = n - r$ as Theorem 2 §5.4 asserts.

As to the image of A, we have $im(A) = col(A)$ again by Theorem 2 §5.4, so $dim[im(A)] = dim[col(A)] = 2$, and a basis is presented in part (1).

2. *Let* $\mathcal{B} = \{(2, 1, 1, -1), (1, -3, 1, 0), (-1, 3, 10, 11), (1, 0, -1, 1)\}$ *in* $\mathbb{R}^4$.

(1) *Show that* $\mathcal{B}$ *is a basis of* $\mathbb{R}^4$. 4 marks

(2) *Express* $X = (a, b, c, d)$ *as a linear combination of the vectors in* $\mathcal{B}$. 4 marks

Solution. Write $X_1 = (2, 1, 1, -1)$, $X_2 = (1, -3, 1, 0)$, $X_3 = (-1, 3, 10, 11)$ and $X_4 = (1, 0, -1, 1)$.

(1). The set $\mathcal{B}$ is orthogonal (for example $X_1 \bullet X_2 = 2-3+1+0 = 0$ and $X_2 \bullet X_3 = -1-9+10+0 = 0$). Hence $\mathcal{B}$ is independent (Theorem 5 §5.3), and so is a basis of $\mathbb{R}^4$ (by Theorem 4 §6.4. Since $|\mathcal{B}| = 4$ and $dim(\mathbb{R}^4) = 4$, $\mathcal{B}$ must be a basis of $\mathbb{R}^4$.

(2). The expansion theorem (Theorem 6 §5.3) gives

$$\begin{aligned} X &= \left(\frac{X \bullet X_1}{\|X_1\|^2}\right) X_1 + \left(\frac{X \bullet X_2}{\|X_2\|^2}\right) X_2 + \left(\frac{X \bullet X_3}{\|X_3\|^2}\right) X_3 + \left(\frac{X \bullet X_4}{\|X_4\|^2}\right) X_4 \\ &= \left(\tfrac{2a+b+c-d}{7}\right) X_1 + \left(\tfrac{a-3b+c}{11}\right) X_2 + \left(\tfrac{-a+3b+10c+11d}{231}\right) X_3 + \left(\tfrac{a-c+d}{3}\right) X_4 \end{aligned}$$

3. *Let A be an $m \times n$ matrix with $rank(A) = n$. If the columns of A are $C_1, C_2, \ldots, C_n$, show that $\mathcal{B} = \{A^T C_1,\ A^T C_2, \ldots, A^T C_n\}$ is a basis of $\mathbb{R}^n$.* 8 marks

Solution. The matrix $A^T A$ is invertible by Theorem 3 §5.4. If we write $A = [C_1\ C_2\ \ldots\ C_n]$ in terms of its columns, then

$$A^T A = A^T[C_1\ C_2\ \cdots\ C_n] = [A^T C_1\ A^T C_2\ \cdots\ A^T C_n].$$

Thus $\mathcal{B}$ consists of the columns of the invertible matrix $A^T A$. By Theorem 3 §5.3, $\mathcal{B}$ is a basis of $\mathbb{R}^n$.

4. *Let A and B denote $n \times n$ matrices. Write $A \sim B$ to mean that A is similar to B.*

(1) *If A is invertible, show that $AB \sim BA$.* 2 marks

(2) *Let $A = rI_n$ where $r \in \mathbb{R}$. If $B \sim A$, show that $B = A$.* 3 marks

(3) *If $A \sim B$ and $A^3 = A$, show that $B^3 = B$.* 3 marks

Solution. (1). $A^{-1}(AB)A = BA$.

(2). Let $B = P^{-1}AP$ where P is invertible. Then $B = P^{-1}(rI_n)P = rP^{-1}P = rI_n = A$.

(3). Let $B = P^{-1}AP$ where P is invertible. Then

$$B^3 = (P^{-1}AP)(P^{-1}AP)(P^{-1}AP) = P^{-1}AI_nAI_nAP = P^{-1}A^3P = P^{-1}AP = B.$$

5. *Let A be an $n \times n$ diagonalizable matrix with eigenvalues $\lambda_1,\ \lambda_2, \ldots, \lambda_n$ (including multiplicities).*

(1) *Show that $det(A) = \lambda_1 \lambda_2 \ldots \lambda_n$.* 4 marks

(2) *Show that $tr(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$ (where $tr(A)$ denotes the trace of A).* 4 marks

Solution. Since A is diagonalizable, there is an invertible matrix P such that $P^{-1}AP = D$ is diagonal. Moreover, $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the diagonal matrix with the λ_i down the main diagonal and zeros elsewhere.

(1). We have $det(P^{-1}AP) = \dfrac{1}{det(P)} det(A) det(P) = det(A)$. That is $det(A) = det(D) = \lambda_1 \lambda_2 \cdots \lambda_n$.

(2). Using Lemma 1 §5.5, we have $tr(P^{-1}AP) = tr[P^{-1}(AP)] = tr[(AP)P^{-1}] = tr(A)$.

Thus $tr(A) = tr(D) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Total: 40 marks