SOLUTIONS to ASSIGNMENT 4.

MATH 311 **FALL 2010**

1. *Let $T : V \to W$ be a linear transformation, and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be vectors in V.*

   (1) *If $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_k)\}$ is independent in W, show that $\mathcal{B}$ is independent in V.* 4 marks

   (2) *If T is onto and $\mathcal{B}$ spans V, show that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_k)\}$ spans W.* 4 marks

   Solution. (1). Let $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0}$, $r_i \in \mathbb{R}$; we must show that each $r_i = 0$. Apply T :

   $$\mathbf{0} = T(\mathbf{0}) = T\left(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k\right) = r_1T\left(\mathbf{v}_1\right) + r_2T(\mathbf{v}_2) + \cdots + r_kT(\mathbf{v}_k)\,.$$

   Hence each $r_i = 0$ because $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_k)\}$ is independent.

   (2). Let $\mathbf{w} \in W$. Since T is onto, let $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v} \in V$. Since $V = span\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ by hypothesis, write $\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_k\mathbf{v}_k$ where each $r_i \in \mathbb{R}$. Because T is linear, this gives $\mathbf{w} = T(\mathbf{v}) = T(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k) = r_1T\left(\mathbf{v}_1\right) + r_2T(\mathbf{v}_2) + \cdots + r_kT(\mathbf{v}_k)$, as required.

2. *Given $a \in \mathbb{R}$, define the evaluation transformation $E_a : \mathbb{P}_n \to \mathbb{R}$ by $E_a[p(x)] = p(a)$ for all polynomials $p(x)$ in $\mathbb{P}_n$. You may assume that E_a is linear.*

   (1) *Show that E_a satisfies $E_a(x^k) = [E_a(x)]^k$ for each $k = 0, 1, 2, \dots$. [Note: $x^0 = 1$.]* 4 marks

   (2) *If $T : \mathbb{P}_n \to \mathbb{R}$ is any linear transformation satisfying $T(x^k) = [T(x)]^k$ for each $k = 0, 1, 2, \dots,$ show that $T = E_a$ for some $a \in \mathbb{R}$.* 4 marks

   Solution. (1). We have $E_a(x^k) = a^k$ and $E_a(x) = a$. Hence $E_a(x^k) = a^k = [E_a(x)]^k$ for each k.

   (2). Given T as in (2), write $T(x) = a$, so $T(x^k) = a^k$ by (2). Now let $p = p(x) = r_0 + r_1x + \cdots + r_nx^n$ be any polynomial in $\mathbb{P}_n$. Since T is linear, and since $x^0 = 1$, we have:

   $$T(p) = r_0T(x^0) + r_1T(x^1) + \cdots + r_nT(x^n) = r_0a^0 + r_1a^1 + \cdots + r_na^n = p(a) = E_a(p).$$

   Since p was arbitrary in $\mathbb{P}_n$, this shows that $T = E_a$.

3. *Define $T : \mathbb{P}_n \to \mathbb{R}$ by taking $T[p(x)]$ to be the sum of all the coefficients of $p(x)$.*

   (1) *Use the dimension theorem to show that $dim[ker(T)] = n$.* 4 marks

   (2) *Conclude that $\mathcal{B} = \{x - 1,\, x^2 - 1, \cdots, x^n - 1\}$ is a basis of $ker(T)$.* 4 marks

   Solution. (1). T is onto $\mathbb{R}$ because, for example, $T(bx) = b$ for any b in $\mathbb{R}$. Hence $im(T) = \mathbb{R}$, so $dim[im(T)] = dim(\mathbb{R}) = 1$. By the dimension theorem, we obtain

   $$dim[ker(T)] = dim[\mathbb{P}_n] - dim[im(T)] = (n + 1) - 1 = n.$$

   (2). The polynomials in $\mathcal{B}$ clearly lie in $ker(T)$, and they are independent because they have distinct degrees (Example 4 §6.3). Since there are n polynomials in $\mathcal{B}$, and since $dim[ker(T)] = n$ by part (1), it follows that $\mathcal{B}$ is a basis of $\mathbb{P}_n$ (Theorem 4 § 6.4).

4. *Let U be an invertible $n \times n$ matrix, and define. $T : \mathbb{M}_{nn} \to \mathbb{M}_{nn}$* by $T(A) = UA$ for every $A \in \mathbb{M}_{nn}$.

   (1) *Show that T is an isomorphism.* 4 marks

   (2) *Does every isomorphism $S : \mathbb{M}_{22} \to \mathbb{M}_{22}$ arise as in part (1)? Defend your answer.*

   [Hint: *transpose.*] 4 marks

Solution. (1). T is linear because, for all $A, B \in M_{nn}$ and $k \in \mathbb{R}$:

$$T(A+B) = U(A+B) = UA + UB = T(A) + T(B), \text{ and}$$
$$T(kA) = U(kA) = kUA = kT(A).$$

If $A \in ker(T)$ then $UA = 0$ so $A = 0$ because U is invertible. Hence $ker(T) = 0$, so T is one-to-one (Theorem 2 §7.2). If $B \in \mathbb{M}_{nn}$ then $T(U^{-1}B) = U(U^{-1}B) = B$ so T is also onto. Hence T is an isomorphism.

(2). Let $T : \mathbb{M}_{nn} \to \mathbb{M}_{nn}$ be the transpose operator, that is $T(A) = A^T$ for all A in M_{nn}. This is linear because, for all $A, B \in M_{nn}$ and $k \in \mathbb{R}$:

$$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B), \text{ and}$$
$$T(kA) = (kA)^T = kA^T = kT(A).$$

But, even if $n = 2$, T is not given by a matrix as in part (1). Indeed, if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ for *any* 2×2 matrix U (look at the $(2,1)$-entry).

5. *Let $T : V \to W$ be a linear transformation.*

(1) *If T is $1:1$ and $TR = TR_1$ for transformations $R, R_1 : U \to V$, show that $R = R_1$.* 4 marks

(2) *If T is onto and $ST = S_1T$ for transformations $S, S_1 : W \to U$, show that $S = S_1$.* 4 marks

Solution. (1). Let $\mathbf{u} \in U$. Since $TR = TR_1$ as transformations from $U \to W$, we have $(TR)(\mathbf{u}) = (TR_1)(\mathbf{u})$, that is $T[R(\mathbf{u})] = T[R_1(\mathbf{u})]$. Because T is one-to-one, this implies that $R(\mathbf{u}) = R_1(\mathbf{u})$. As this holds for all $\mathbf{u}$ in U, it shows that $R = R_1$.

(2). Given $\mathbf{w}$ in W we have $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{w}$ in V. Since $ST = ST_1 : V \to U$, we get

$$S(\mathbf{w}) = S[T(\mathbf{v})] = (ST)(\mathbf{v}) = (S_1T)(\mathbf{v}) = S_1[T(\mathbf{v})] = S_1(\mathbf{w}).$$

Since $\mathbf{w}$ was arbitrary in W, this shows that $S = S_1$.

Total: 40 marks