SOLUTIONS to ASSIGNMENT 5.

MATH 311 **FALL 2010**

1. #13 §8.1. *If U is a subspace of $\mathbb{R}^n$ show that $U^{\perp\perp} = U$ where $U^{\perp\perp} = (U^{\perp})^{\perp}$.* [Hint: Show that $U \subseteq U^{\perp\perp}$ and use Theorem 4 §8.1 twice.] 8 marks

   Solution. If $X \in U$ then $X \bullet Y = 0$ for all $Y \in U^{\perp}$, whence $X \in U^{\perp\perp}$. Now Theorem 4 §8.1 shows that $dim(U) = n - dim(U^{\perp})$. But the same theorem applied to $U^{\perp\perp}$ shows that $dim(U^{\perp}) = n - dim(U^{\perp\perp})$. Hence $dim(U) = n - (n - dim(U^{\perp\perp}) = dim(U^{\perp\perp})$. Since $U \subseteq U^{\perp\perp}$ it follows that $U = U^{\perp\perp}$.

2. *If A is an $m \times n$ matrix, show that $null(A) = [row(A)]^{\perp}$.* 8 marks

   Solution. Let $R_1, R_2, \ldots, R_m$ denote the rows of A. If $X \in \mathbb{R}^n$, the j^{th} entry of AX is $R_i \bullet X$. Hence $null(A) = \{X \mid AX = 0\} = \{X \mid R_i \bullet X = 0 \text{ for each } i\}$. But $row(A) = span\{R_1, R_2, \ldots, R_m\}$, so Lemma 2 §8.1 shows that $null(A) = [row(A)]^{\perp}$, as desired.

3. #13 §8.1. *Assume that A and B are orthogonally similar (that is $P^TAP = B$ for some orthogonal matrix P).*

   (a) *If A and B are invertible, show that A^{-1} and B^{-1} are orthogonally similar.* 3 marks

   (b) *Show that A^2 and B^2 are orthogonally similar.* 2 marks

   (c) *Show that if A is symmetric, so also is B.* 3 marks

   Solution. Assume $B = P^TAP$ where P is orthogonal.

   (a) Since $P^{-1} = P^T$, we have $B^{-1} = P^{-1}A^{-1}(P^{-1})^T = P^TA^{-1}P$,.

   (b) $B^2 = (P^TAP)(P^TAP) = P^TA^2P$.

   (c) If $A = A^T$ then $B^T = P^TA^TP = P^TAP = B$.

4. #5 §8.3. *If A and B are positive definite, show that $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is positive definite.* 8 marks

   Solution.If A is $n \times n$ and B is $m \times m$, let $Z \neq 0$ in $\mathbb{R}^{m+n}$. If $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ where $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$, then $Z^T \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} Z = [X^T \quad Y^T] \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = X^TAX + Y^TBY > 0$ because $X \neq 0$ or $Y \neq 0$.

5. (a) #9 §8.6. *Show that $\langle ZX, Y \rangle = \langle X, Z^HY \rangle$ for all $Z \in M_{nn}(C)$ and all $X, Y \in \mathbb{C}^n$.* 3 marks

   #15 §8.6. *Let U be a unitary matrix. Show that*

   (b) *$\|UX\| = \|X\|$ for all X in $\mathbb{C}^n$.* 2 marks

   (c) *$|\lambda| = 1$ for every eigenvalue λ of U.* 3 marks

   Solution. (a). $\langle X, Y \rangle = X^T\bar{Y}$ for columns X and Y in $\mathbb{C}^n$ Hence

   $$\langle X, Z^HY \rangle = X^T\overline{(Z^HY)} = X^TZ^T\bar{Y} = (ZX)^T\bar{Y} = \langle ZX, Y \rangle.$$

   (b). Use (a): $\|UX\|^2 = \langle UX, UX \rangle = \langle X, U^HUX \rangle = \langle X, X \rangle = \|X\|^2$.

   (c). If $UX = \lambda X$, $X \neq 0$, then $\|X\| = \|UX\| = \|\lambda X\| = |\lambda|\|X\|$, using (b). Since $\|X\| \neq 0$, this gives $|\lambda| = 1$.

Total: 40 marks