SOLUTIONS to ASSIGNMENT 6.

MATH 311 **FALL 2010**

1. §9.1 #4(c). Let $T : \mathbf{P}_2 \to \mathbb{R}^2$ be given by $T(a+bx+cx^2) = (a+c, 2b)$. Consider the bases $B = \{1, x, x^2\}$ of $\mathbf{P}_2$ and $D = \{(1,0), (1,-1)\}$ of $\mathbb{R}^2$. Find the matrix of T corresponding to these bases, use it to compute $C_D[T(\mathbf{v})]$, and and use that to compute $T(\mathbf{v})$ where $\mathbf{v} = a + bx + cx^2$. 8 marks

   Solution. Compute $M_{DB}(T)$ directly:
$$M_{DB}(T) = \left[C_D[T(1)]\ \ C_D[T(x)]\ \ C_D[T(x^2)]\right] = \left[C_D[(1,0)]\ \ C_D[(0,2)]\ \ C_D[(1,0)]\right] = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \end{bmatrix}.$$
   Hence Theorem 2 §9.1 gives
$$C_D[T(\mathbf{v})] = M_{DB}(T)C_B(\mathbf{v}) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+2b+c \\ -2b \end{bmatrix}.$$
   Finally we use this to recover the action of T :
$$T(\mathbf{v}) = (a+2b+c)(1,0) + (-2b)(1,-1) = (a+c, 2b)$$
   as expected.

2. §9.1 #14. Let U be an invertible $n \times n$ matrix, and let $D = \{\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n\}$ where $\mathbf{f}_j$ is column j of U for each j. If B is the standard basis of $\mathbb{R}^n$, show that $M_{BD}(1_{\mathbb{R}^n}) = U$. 8 marks

   Solution. Since B is the standard basis, we have $C_B(\mathbf{v}) = \mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^n$. Since B is the standard basis of $\mathbb{R}^{n,}$ we have $C_B(X) = X$ for all $X \in \mathbb{R}^n$. Hence:
$$M_{BD}(1_{\mathbb{R}^n}) = [C_B(\mathbf{f}_1) \cdots C_B(\mathbf{f}_n)] = [\mathbf{f}_1 \cdots \mathbf{f}_n] = U.$$

3. §9.2 #15. Let $B = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ be any ordered basis of $\mathbb{R}^n$, written as columns. Define the matrix $Q = [\mathbf{b}_1\ \ \mathbf{b}_2\ \ \cdots\ \ \mathbf{b}_n]$ with the $\mathbf{b}_i$ as its columns. Show that $QC_B(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$.
   8 marks

   Solution. If $\mathbf{v} = v_1\mathbf{b}_1 + \cdots + v_n\mathbf{b}_n$ where each $v_i \in \mathbb{R}$, then block multiplication gives
$$QC_B(\mathbf{v}) = [\mathbf{b}_1\ \ldots\ \mathbf{b}_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{b}_1 + \cdots + v_n\mathbf{b}_n = \mathbf{v}.$$

4. §9.3 #16(a). Let $V \xrightarrow{T} W \xrightarrow{S} V$ be linear transformations, and assume that $dim(V)$ and $dim(W)$ are finite. If $ST = 1_V$, show that $W = im(T) \oplus ker(S)$.
   [Hint: If $\mathbf{w} \in W$, first show that $\mathbf{w} - TS(\mathbf{w}) \in ker(S)$.] 8 marks

   Solution. As per the Hint: Since $ST = 1_V$, we have
$$S[\mathbf{w} - TS(\mathbf{w})] = S(\mathbf{w}) - ST(S\mathbf{w})] = S(\mathbf{w}) - S(\mathbf{w}) = 0.$$
   Thus $\mathbf{w} - TS(\mathbf{w})$ lies in $\ker S$ for all $\mathbf{w}$, so $\mathbf{w} = [\mathbf{w} - TS(\mathbf{w})] + T[S(\mathbf{w})]$ lies in $\ker\ S + im(T)$. It follows that $W = ker(S) + im(T)$.

   If $\mathbf{w}$ is in $ker(S) \cap im(T)$, write $\mathbf{w} = T(\mathbf{v})$, $\mathbf{v}$ in V. Then $\mathbf{v} = ST(\mathbf{v}) = S(\mathbf{w}) = \mathbf{0}$, so $\mathbf{w} = T(\mathbf{v}) = T(\mathbf{0}) = \mathbf{0}$. This means that $ker(S) \cap im(T) = 0$.

5. §9.3 #8(a). Define a linear transformation $T : \mathbf{P}_2 \to \mathbf{P}_2$ by

$$T(a + bx + cx^2) = (-a + 2b + c) + (a + 3b + c)x + (a + 4b)x^2.$$

If $U = span\{1, x + x^2\}$, show that U is T-invariant, find a block upper triangular matrix for T, and use that to compute the characteristic polynomial $c_T(x)$ of T. 8 marks

Solution. $T(1) = -1 + (x + x^2)$ is in U, and $T(x + x^2) = 3 + 4x + 4x^2 = 3 + 4(x + x^2)$ is in U. Hence U is T-invariant by Example 3. If $B = \{1, x + x^2, x^2\}$ then

$$M_B(T) = \left[C_D[T(1)] \;\; C_D[T(x)] \;\; C_D[T(x^2)]\right] = \begin{bmatrix} -1 & 3 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Hence $c_T(x) = det \begin{bmatrix} x+1 & -3 \\ -1 & x-4 \end{bmatrix} det[x + 1] = (x^2 - 3x - 7)(x + 1)$.

Total: 40 marks