UNIVERSITY OF CALGARY

DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 321 —- Quiz no.4—- Sections T05/T07 —- Dec. 3, 2008

TIME: 50 minutes

Marks

NAME: Marking Key

10 1. A union official wanted to get an idea of whether a majority of workers at a large corporation would favor a contract proposal. She surveyed 500 randomly-selected workers and found that 265 of them favored the proposal.

6 (a) Find a 95% confidence interval for the proportion, p, of all the workers who favor the contract proposal.

point est of p is $\hat{p} = \frac{X}{n} = \frac{265}{500} = 0.53$ ← 2

$100(1-\alpha) = 95\% \Rightarrow \alpha = .05$, so $z_{\alpha/2} = z_{.025} = 1.960$ ← 1

The 95% C.I. is $\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ ← 1

or $0.53 \pm 1.96\sqrt{\frac{(.53)(.47)}{500}}$ ← 1

or $0.53 \pm 1.96(.02232)$ or 0.53 ± 0.04375 1

or $0.4863 < p < 0.5738$

4 (b) Test the null hypothesis $H_0 : p = 0.5$ against the alternative hypothesis $H_1 : p > 0.5$, using significance level $\alpha = 0.10$.

$H_0: p = 0.5$

$H_1: p > 0.5$

$\alpha = .10$

Reject H_0 if $z = \frac{\hat{p} - 0.5}{\sqrt{\frac{(0.5)(0.5)}{500}}} > z_{.10} = 1.282$ 2 1

$z = \frac{0.53 - 0.50}{.02236} = 1.342 > 1.282$,

so we reject H_0 and conclude, at level $\alpha = .10$, that $p > 0.5$. 1

ID number: ______________________

Marks

10

2. In a study to determine whether a certain stimulant produces hyperactivity, 60 mice were injected with a standard dose of the stimulant. Afterward, each mouse was given a hyperactivity score. For the 60 scores, the sample mean was 15.3 and the sample standard deviation was 2.7.

6 (a) Compute a 90% confidence interval for the population mean score, μ.

$n=60 \quad \bar{x}=15.3 \quad s=2.7$ (2)

$100(1-\alpha)=90 \Rightarrow \alpha=.10$, so $z_{\alpha/2}=z_{.05}=1.645$

90% C.I. is $\bar{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}}$

OR $15.3 \pm 1.645\frac{2.7}{\sqrt{60}}$ ← (1) OR $\underbrace{15.3 \pm 1.645(0.3486)}_{(1)}$

OR 15.3 ± 0.5734 ← (1)

OR $14.73 < \mu < 15.87$ ← (1)

4 (b) Test the null hypothesis $H_0 : \mu = 15.0$ against the alt. hypothesis $H_1 : \mu > 15.0$. Use significance level $\alpha = .05$.

$H_0: \mu=15$

$H_1: \mu>15$

$\alpha=.05$

(1) (1)

Reject H_0 if $z=\frac{\bar{x}-15}{s/\sqrt{60}} > z_{.05}=1.645$

(1)

$z=\frac{15.3-15.0}{2.7/\sqrt{60}}=\frac{0.3}{0.3486}=.8605<1.645$

So do not reject H_0 at level $\alpha=.05$ & conclude $H_0: \mu=15.0$ (1)

Marks

20 3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. Let

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

To work this problem, you may make use of the fact that the random variable $(n-1)S^2/\sigma^2$ is known to have a Chi-square distribution, with mean $n-1$ and variance $2(n-1)$.

5 (a) Show that S^2 is an unbiased estimator of σ^2.

$$E(S^2) = E\left[\frac{\sigma^2}{n-1}\frac{n-1}{\sigma^2}S^2\right] = \frac{\sigma^2}{n-1}E\left[\frac{(n-1)S^2}{\sigma^2}\right] = \frac{\sigma^2}{n-1}(n-1) = \sigma^2$$

2 1 1 1

5 (b) Find the MSE of S^2.

Since $ES^2 = \sigma^2$, $MSE(S^2) = V(S^2)$ 2

$$= V\left(\frac{\sigma^2}{n-1}\frac{n-1}{\sigma^2}S^2\right) = \left(\frac{\sigma^2}{n-1}\right)^2 V\left(\frac{n-1}{\sigma^2}S^2\right) = \frac{\sigma^4}{(n-1)^2}2(n-1)$$

1 1

$$= \frac{2\sigma^4}{n-1}$$ 1

5 (c) Consider the statistic $\hat{\sigma}^2$, defined by

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Show that $\hat{\sigma}^2$ is a biased estimator of σ^2, and find the bias of $\hat{\sigma}^2$.

$$E(\hat{\sigma}^2) = E\left[\frac{n-1}{n}\frac{1}{n-1}\sum(X_i-\overline{X})^2\right] = E\left[\frac{n-1}{n}S^2\right] = \frac{n-1}{n}ES^2 = \frac{n-1}{n}\sigma^2$$

1 2

$\neq \sigma^2$, so $\hat{\sigma}^2$ is biased.

The bias is $E(\hat{\sigma}^2) - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = \sigma^2\left(\frac{n-1}{n} - \frac{n}{n}\right)$

1

$= -\frac{1}{n}\sigma^2$ 1

5 (d) Find the MSE of $\hat{\sigma}^2$.

$MSE(\hat{\sigma}^2) = Bias^2(\hat{\sigma}^2) + V(\hat{\sigma}^2)$ 2

$$V(\hat{\sigma}^2) = V\left(\frac{n-1}{n}S^2\right) = \left(\frac{n-1}{n}\right)^2 V(S^2) = \frac{(n-1)^2}{n^2}\frac{2\sigma^4}{n-1} = \frac{2(n-1)}{n^2}\sigma^4$$

1 1 1

Total Marks = 40

UNIVERSITY OF CALGARY

DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 321 — Quiz no.4— Section T06 — Dec. 3, 2008

TIME: 50 minutes

NAME: Marking Key

Marks

10 1. A union official wanted to get an idea of whether a majority of workers at a large corporation would favor a contract proposal. She surveyed 400 randomly-selected workers and found that 220 of them favored the proposal.

6 (a) Find a 90% confidence interval for the proportion, p, of all the workers who favor the contract proposal.

Point est of p is $\hat{p} = \frac{X}{n} = \frac{220}{400} = 0.55$ ← 2

$100(1-\alpha) = 90 \Rightarrow \alpha = .10$, so $z_{\alpha/2} = z_{.05} = 1.645$ ← 1

The 90% C.I. for p is $\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ ← 1

~~OR 0.55 ± 1.645(.02487) OR 0.55~~

OR $0.55 \pm 1.645\sqrt{.55(.45)/400}$ ← 1

OR $0.55 \pm 1.645(.02487)$ OR $0.55 \pm .0409$

OR $0.5091 < p < 0.5909$ ← 1

4 (b) Test the null hypothesis $H_0 : p = 0.5$ against the alternative hypothesis $H_1 : p > 0.5$, using significance level $\alpha = 0.05$.

$H_0 : p = 0.5$

$H_1 : p > 0.5$

$\alpha = .05$

Reject H_0 if $z = \frac{\hat{p} - 0.5}{\sqrt{\frac{(0.5)(0.5)}{400}}} > z_{.05} = 1.645$ 2 1

$z = \frac{0.55 - 0.50}{0.025} = 2 > 1.645$, so we reject H_0 and conclude that $p > 0.5$. 1

ID number:

Marks

10 2. In a study to determine whether a certain stimulant produces hyperactivity, 40 mice were injected with a standard dose of the stimulant. Afterward, each mouse was given a hyperactivity score. For the 40 scores, the sample mean was 15.7 and the sample standard deviation was 3.8.

6 (a) Compute a 95% confidence interval for the population mean score, μ.

$n = 40 \quad \bar{x} = 15.7 \quad s = 3.8$

$100(1-\alpha) = 95$, so $\alpha = .05$, so $z_{\alpha/2} = z_{.025} = 1.96$ ← 2

95% C.I. is $\bar{x} \pm z_{\alpha/2} \frac{s}{\sqrt{n}}$ 1

OR $15.7 \pm 1.96 \frac{3.8}{\sqrt{40}}$ OR $\underbrace{15.7 \pm 1.96(0.6008)}_{1}$

OR 15.7 ± 1.178 ← 1

OR $14.52 < \mu < 16.88$ ← 1

4 (b) Test the null hypothesis $H_0 : \mu = 15.0$ against the alt. hypothesis $H_1 : \mu > 15.0$. Use significance level $\alpha = 0.10$.

$H_0: \mu = 15$
$H_1: \mu > 15$
$\alpha = .10$

Reject H_0 if $z = \frac{\bar{x} - 15}{s/\sqrt{40}} > z_{.10} = 1.282$ 1 1

$z = \frac{15.7 - 15}{3.8/\sqrt{40}} = \frac{0.7}{0.6008} = 1.165 < 1.282,$ 1

so do not reject H_0, & conclude $H_0: \mu = 15.$ ← 1

Marks

20 3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. Let

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

To work this problem, you may make use of the fact that the random variable $(n-1)S^2/\sigma^2$ is known to have a Chi-square distribution, with mean $n-1$ and variance $2(n-1)$.

5 (a) Show that S^2 is an unbiased estimator of σ^2.

$$E(S^2) = E\left[\frac{\sigma^2}{n-1}\,\frac{n-1}{\sigma^2}\,S^2\right] = \frac{\sigma^2}{n-1}\,E\left[\frac{(n-1)S^2}{\sigma^2}\right] = \frac{\sigma^2}{n-1}(n-1) = \sigma^2.$$

2 1 2 1

5 (b) Find the MSE of S^2.

Since $ES^2 = \sigma^2$, $MSE(S^2) = V(S^2)$ 2

$$= V\left(\frac{\sigma^2}{n-1}\,\frac{n-1}{\sigma^2}\,S^2\right) = \left(\frac{\sigma^2}{n-1}\right)^2 V\left(\frac{n-1}{\sigma^2}\,S^2\right) = \frac{\sigma^4}{(n-1)^2}\,2(n-1)$$

1 1

$$= \frac{2\sigma^4}{n-1}$$ 1

5 (c) Consider the statistic $\hat{\sigma}^2$, defined by

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Show that $\hat{\sigma}^2$ is a biased estimator of σ^2, and find the bias of $\hat{\sigma}^2$.

$$E(\hat{\sigma}^2) = E\left[\frac{n-1}{n}\,\frac{1}{n-1}\sum(X_i-\overline{X})^2\right] = E\left[\frac{n-1}{n}\,S^2\right] = \frac{n-1}{n}\,ES^2 = \frac{n-1}{n}\sigma^2$$

1 2

$\neq \sigma^2$, so $\hat{\sigma}^2$ is biased.

The bias is $E(\hat{\sigma}^2) - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = \sigma^2\left(\frac{n-1}{n} - \frac{n}{n}\right)$

1

$$= -\frac{1}{n}\sigma^2$$ 1

5 (d) Find the MSE of $\hat{\sigma}^2$.

$MSE(\hat{\sigma}^2) = Bias^2(\hat{\sigma}^2) + V(\hat{\sigma}^2)$ 2

$$V(\hat{\sigma}^2) = V\left(\frac{n-1}{n}\,S^2\right) = \left(\frac{n-1}{n}\right)^2 V(S^2) = \frac{(n-1)^2}{n^2}\,\frac{2\sigma^4}{n-1} = \frac{2(n-1)}{n^2}\sigma^4$$

1 1 1

Total Marks = 40