

UNIVERSITY OF CALGARY

DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 323, Section L01—Midterm Exam — March 2, 2007

TIME: 50 minutes

MarksNAME: Solutions

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1. Random variables X and Y have the following joint density function:

$$f(x, y) = \begin{cases} Cx/y^2 & \text{if } 1 \leq x \leq 2, 1 \leq y \leq x \\ 0 & \text{elsewhere.} \end{cases}$$

Find:

- 2 (a) the value of the constant
- C
- ;

$$1 = C \int_1^2 \int_{y=1}^{\infty} \frac{x}{y^2} dy dx = C \int_1^2 (x-1) dx = \frac{1}{2} C$$

$$\text{so } \underline{\underline{C = 2}}$$

- 3 (b) the marginal density function of
- X
- ;

$$f(x) = \int_{y=1}^x \frac{2x}{y^2} dy = 2x \left(1 - \frac{1}{x}\right) = 2(x-1), \quad \begin{array}{l} \text{for } 1 < x < 2 \\ \text{elsewhere} \end{array}$$

- 3 (c) the marginal density function of
- Y
- ;

$$f(y) = \int_{x=y}^2 \frac{2x}{y^2} dx = \frac{1}{y^2} (4-y^2) = \left(\frac{4}{y^2} - 1\right), \quad \begin{array}{l} \text{for } 1 < y < 2 \\ \text{elsewhere} \end{array}$$

- 2 (d) the conditional density function of
- X
- given that
- $Y = 3/2$
- .

$$f(x, y=3/2) = \frac{f(x, \frac{3}{2})}{f_y(\frac{3}{2})} = \frac{2x \cdot \frac{4}{9}}{4(\frac{4}{9}) - 1} = \left(\frac{8}{7}x\right), \quad \begin{array}{l} \text{for } \frac{3}{2} < x < 2 \\ \text{elsewhere} \end{array}$$

- 10 2. Random variables X and Y have joint density function

$$f(x, y) = \begin{cases} e^{-x-y} & \text{if } x > 0, y > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Let $W = X/Y$. Find the density function of W .

Make a 2-dim. change-of-variables $w = \frac{x}{y}$, $z = y$,
which is 1-1 from $\mathbb{R}^+ \times \mathbb{R}^+$ onto $\mathbb{R}^+ \times \mathbb{R}^+$ with inverse

$$\text{So } J = \begin{vmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} z & w \\ 0 & 1 \end{vmatrix} = z. \quad \leftarrow (2)$$

$$\text{So } f(w, z) = e^{-wz-z} = e^{-z(w+1)} \quad , \quad z > 0, w > 0 \\ = 0 \quad \text{otherwise} \quad \leftarrow (2)$$

$$\text{So } f(w) = \left(\int_0^\infty z e^{-z(w+1)} dz \right) = \int_0^\infty z d \left[-\frac{1}{w+1} e^{-z(w+1)} \right] \\ = -z \frac{1}{w+1} e^{-z(w+1)} \Big|_0^\infty + \frac{1}{w+1} \int_0^\infty e^{-z(w+1)} dz \\ = \frac{1}{(w+1)^2} \quad , \quad w > 0 \\ = 0 \quad , \quad \text{otherwise} \quad \leftarrow (2)$$

[An alternative solution (which of course gives the same answer), starts with the change of variable $x = wz$, $y = x$.]

- 10 3. Let X_1, X_2, X_3 be independent identically distributed random variables, each having the following density function:

$$f(x) = \begin{cases} 2x/\theta^2 & \text{if } 0 < x < \theta \\ 0 & \text{elsewhere,} \end{cases}$$

where $\theta > 0$.

- 4 (a) Find the density function of $X_{(3)} = \max\{X_1, X_2, X_3\}$.

$$F_{X_{(3)}}(x) = [F_{X_1}(x)]^3 = \left(\frac{x^2}{\theta^2}\right)^3 = \frac{x^6}{\theta^6}, \text{ for } 0 < x < \theta$$

$$\text{so } f_{X_{(3)}}(x) = F'_{X_{(3)}}(x) = \frac{6x^5}{\theta^6}, \text{ for } 0 < x < \theta$$

$$= 0 \quad \text{elsewhere}$$

- 3 (b) Find the expected value of $X_{(3)}$.

$$E(X_{(3)}) = \int_0^\theta x \cdot \frac{6x^5}{\theta^6} dx = \frac{1}{\theta^6} \frac{6}{7} \theta^7 = \frac{6}{7} \theta \quad \text{← (3)}$$

- 3 (c) Find a function of $X_{(3)}$ which is an unbiased estimator of θ .

By inspection, $T(X_{(3)}) = \frac{7}{6} X_{(3)}$ should be unbiased.

check: $E(T) = \frac{7}{6} E(X_{(3)}) = \frac{7}{6} \frac{6}{7} \theta = \theta. \checkmark$

- 10 4. Let Z be a random variable with the standard normal density function $f(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$ for $-\infty < z < \infty$. Let $Q = Z^2$. Derive the density function of Q .

For $q > 0$,

$$\begin{aligned} F(q) &= P[Q \leq q] = P[Z^2 \leq q] = P[-\sqrt{q} \leq Z \leq \sqrt{q}] \\ &= F(\sqrt{q}) - F(-\sqrt{q}) \end{aligned}$$

$$\begin{aligned} \text{So } f(q) &= F'(q) = f(\sqrt{q}) \frac{1}{2} q^{-1/2} - f(-\sqrt{q}) (-\frac{1}{2} q^{-1/2}) \\ &= f(\sqrt{q}) q^{-1/2} = \frac{1}{\sqrt{2\pi}} e^{-q/2} \cdot q^{-1/2} \\ \text{since } f(\sqrt{q}) &= f(-\sqrt{q}), \quad q > 0. \end{aligned}$$

Note: 5 marks for a formal (but wrong) application of the change-of-variable formula for 1-1 mappings, leading to $1/2$ of the right answer.

3 marks for noting that it is known that $Z^2 \sim \chi_1^2$ and writing down the χ_1^2 -density. (a derivation of the fact that $Z^2 \sim \chi_1^2$ was called for.)

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5. Let X_1, X_2, X_3 denote a random sample of size 3 from a uniform distribution on $[0,1]$.
 Let $X_{(1)}, X_{(2)}, X_{(3)}$ denote the corresponding order statistics.
 Compute the covariance of $X_{(1)}$ and $X_{(3)}$.

$$\text{cov}(X_{(1)}, X_{(3)}) = E[X_{(1)}X_{(3)}] - E(X_{(1)})E(X_{(3)})$$

$$f_{X_{(1)}}(x) = 3(1-x)^2 \text{ for } 0 < x < 1,$$

$$\therefore E(X_{(1)}) = \int_0^1 3x(1-x)^2 dx = \frac{1}{4} \quad \leftarrow (2)$$

$$f_{X_{(3)}}(x) = 3x^2, \quad 0 < x < 1,$$

$$\therefore E(X_{(3)}) = \int_0^1 3x^3 dx = \frac{3}{4} \quad \leftarrow (2)$$

$$f_{X_{(1)}, X_{(3)}}(x_1, x_3) = \frac{3!}{1!1!1!} f(x_1) f(x_3) [F(x_3) - F(x_1)] = 6(x_3 - x_1),$$

$$0 < x_1 < x_3 < 1 \quad \leftarrow (2)$$

$$\therefore E[X_{(1)}X_{(3)}] = \int_{x_3=0}^1 \int_{x_1=0}^{x_3} 6x_1x_3(x_3 - x_1) dx_1 dx_3 \quad \leftarrow (1)$$

$$= \int_{x_3=0}^1 6x_3 \int_0^{x_3} (x_1x_3 - x_1^2) dx_1 dx_3$$

$$= \int_0^1 6x_3 \left[\frac{1}{2}x_3^3 \right] dx_3 = \int_0^1 x_3^4 dx_3 = \frac{1}{5} \quad \leftarrow (1)$$

$$\therefore \text{cov}(X_{(1)}, X_{(3)}) = \frac{1}{5} - \frac{1}{4} \cdot \frac{3}{4} = \underline{\underline{\frac{1}{20}}}$$