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# MATH 323

## Solutions to Assignment #1

- 5.2 The sample space for the toss of three balanced coins, the values for  $Y_1$  and  $Y_2$  at each outcome, and the probability of each outcome are given below:

OUTCOMES	( $y_1, y_2$ )	PROBABILITY
$HHH$	(3, 1)	$\frac{1}{8}$
$HHT$	(3, 1)	$\frac{1}{8}$
$HTH$	(2, 1)	$\frac{1}{8}$
$HTT$	(1, 1)	$\frac{1}{8}$
$THH$	(2, 2)	$\frac{1}{8}$
$THT$	(1, 2)	$\frac{1}{8}$
$TTH$	(1, 3)	$\frac{1}{8}$
$TTT$	(0, -1)	$\frac{1}{8}$

	$y_1$	0	1	2	3
$y_2$	0	$\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{2}{8}$
-1	$\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
1	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	0
2	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	0
3	0	$\frac{1}{8}$	0	0	0

b.  $F(2, 1) = P(Y_1 \leq 2, Y_2 \leq 1) = p(0, -1) + p(1, 1) + p(2, 1) = \frac{1}{8} + \frac{1}{8} + \frac{2}{8} = \frac{1}{2}$

- 5.6 a. We must have

$$F(\infty, \infty) = \int_0^1 \int_0^1 Ky_1 y_2 dy_1 dy_2 = 1.$$

Then

$$\int_0^1 \int_0^1 Ky_1 y_2 dy_1 dy_2 = K \int_0^1 (y_2) \left[ \frac{y_1^2}{2} \right]_0^1 dy_2 = \frac{K}{2} \int_0^1 y_2 dy_2 = \frac{K}{2} \left[ \frac{y_2^2}{2} \right]_0^1 = \frac{K}{4} = 1$$

so that  $K = 4$ .

b.  $F(y_1, y_2) = \int_0^{y_2} \int_0^{y_1} 4t_1 t_2 dt_1 dt_2 = \int_0^{y_2} \left[ \frac{4t_1^2}{2} \right]_0^{y_1} dt_2 = \int_0^{y_2} 2y_1^2 t_2 dt_2 = y_1^2 y_2^2$

for  $0 \leq y_1 \leq 1$  and  $0 \leq y_2 \leq 1$ . Recall that

$$F(y_1, y_2) = \begin{cases} 0, & \text{for } y_1 \leq 0 \text{ or } y_2 \leq 0 \\ 1, & \text{for } y_1 \geq 1 \text{ and } y_2 \geq 1. \end{cases}$$

c.  $P(Y_1 \leq \frac{1}{2}, Y_2 \leq \frac{3}{4}) = F\left(\frac{1}{2}, \frac{3}{4}\right) = \left(\frac{1}{2}\right)^2 \left(\frac{3}{4}\right)^2 = \frac{9}{64}$

- 5.10** The region over which the joint density function is positive is the triangular region shown in Figure 5.5. The shaded area is the region in which  $Y_1 \leq \frac{3}{4}$  and  $Y_2 \leq \frac{3}{4}$ .

$$\begin{aligned} \text{a. } P(Y_1 \leq \frac{3}{4}, Y_2 \leq \frac{3}{4}) &= \int_{0}^{3/4} \int_{0}^{1-y_1} 2 dy_1 dy_2 + \int_{1/4}^{3/4} \int_{0}^{1-y_1} 2 dy_2 dy_1 \\ &= \frac{3}{8} + 2 \int_{1/4}^{3/4} (1 - y_1) dy_1 \\ &= \frac{3}{8} + 2 \left[ y_1 - \frac{y_1^2}{2} \right]_{1/4}^{3/4} = \frac{7}{8} \end{aligned}$$

$$\text{b. } P(Y_1 \leq \frac{1}{2}, Y_2 \leq \frac{1}{2}) = \int_{0}^{1/2} \int_{0}^{1/2} 2 dy_1 dy_2 = \frac{1}{2}$$

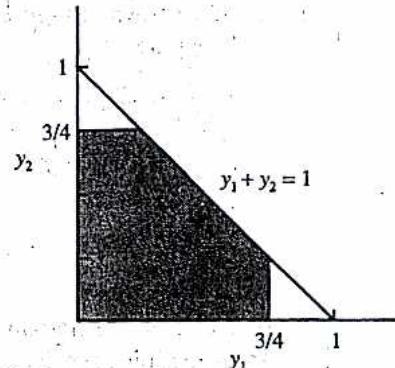


Figure 5.5

$$\text{5.14 a. } P(Y_1 < \frac{1}{2}, Y_2 > \frac{1}{4}) = \int_{1/4}^1 \int_0^{1/2} (y_1 + y_2) dy_1 dy_2 = \int_{1/4}^1 \left( \frac{1}{8} + \frac{y_2}{2} \right) dy_2 = \frac{21}{64}$$

b. Refer to Figure 5.8. Integrating over the shaded region, we obtain

$$\begin{aligned} P(Y_1 + Y_2 \leq 1) &= \int_0^1 \int_0^{1-y_2} (y_1 + y_2) dy_1 dy_2 \\ &= \int_0^1 \left( \frac{1}{2} - \frac{y_2^2}{2} \right) dy_2 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}. \end{aligned}$$

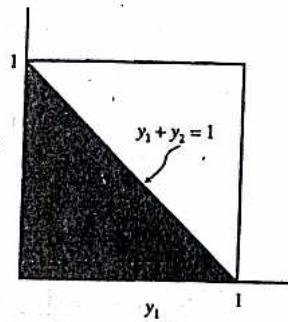


Figure 5.8

5.22 a. By definition,

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 = \int_0^1 4y_1 y_2 dy_2 = (4y_1) \left(\frac{y_2^2}{2}\right) \Big|_0^1 = 2y_1 \quad \text{for } 0 \leq y_1 \leq 1$$

and

$$f_2(y_2) = \int_0^1 4y_1 y_2 dy_1 = (4y_2) \left(\frac{y_1^2}{2}\right) \Big|_0^1 = 2y_2 \quad \text{for } 0 \leq y_2 \leq 1$$

b. By the definition of conditional probability,

$$P(Y_1 \leq \frac{1}{2} | Y_2 > \frac{3}{4}) = \frac{P(Y_1 \leq \frac{1}{2}, Y_2 > \frac{3}{4})}{P(Y_2 > \frac{3}{4})}.$$

Now

$$\begin{aligned} P(Y_1 \leq \frac{1}{2}, Y_2 > \frac{3}{4}) &= \int_0^{1/2} \int_{3/4}^1 4y_1 y_2 dy_2 dy_1 = \int_0^{1/2} 2y_1 [y_2^2]_{3/4}^1 dy_1 = \frac{7}{16} y_1^2 \Big|_0^{1/2} \\ &= \frac{7}{64} \end{aligned}$$

and

$$P(Y_2 > \frac{3}{4}) = \int_{3/4}^1 f_2(y_2) dy_2 = \int_{3/4}^1 2y_2 dy_2 = y_2^2 \Big|_{3/4}^1 = \frac{7}{16}.$$

Hence

$$P(Y_1 \leq \frac{1}{2} | Y_2 > \frac{3}{4}) = \frac{\left(\frac{7}{64}\right)}{\left(\frac{7}{16}\right)} = \frac{1}{4}.$$

Notice this is the same probability as  $P(Y_1 \leq \frac{1}{2})$ .

c. By Definition 5.7, if  $0 < y_2 \leq 1$

$$f(y_1 | y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{4y_1 y_2}{2y_2} = 2y_1, \quad 0 \leq y_1 \leq 1.$$

Notice this is the same as  $f(y_1)$ .

d. If  $0 < y_1 \leq 1$ ,

$$f(y_2 | y_1) = \frac{f(y_1, y_2)}{f_1(y_1)} = \frac{4y_1 y_2}{2y_1} = 2y_2, \quad 0 \leq y_2 \leq 1.$$

Notice this is the same as  $f(y_2)$

$$e. P(Y_1 \leq \frac{3}{4} | Y_2 = \frac{1}{2}) = \int_0^{3/4} f(y_1 | y_2 = \frac{1}{2}) dy_1 = \int_0^{3/4} 2y_1 dy_1 = y_1^2 \Big|_0^{3/4} = \frac{9}{16}$$

$$5.26 a. P(Y_1 \geq \frac{1}{2}, Y_2 \leq \frac{1}{4}) = \int_0^{1/4} \int_{1/2}^{1-y_2} 2 dy_1 dy_2 = \int_0^{1/4} 2 \left(1 - y_2 - \frac{1}{2}\right) dy_2 + [y_2 - y_2^2]_0^{1/4} \\ = \frac{3}{16}$$

and

$$P(Y_2 \leq \frac{1}{4}) = \int_0^{1/4} 2(1 - y_2) dy_2 = \frac{1}{2} - \frac{1}{16} = \frac{7}{16}$$

Hence

$$P(Y_1 \geq \frac{1}{2} | Y_2 \leq \frac{1}{4}) = \frac{\left(\frac{3}{16}\right)}{\left(\frac{7}{16}\right)} = \frac{3}{7}.$$

Notice we could have this probability without integration as the joint density is constant.

b. If  $0 \leq y_2 < 1$ , the conditional distribution of  $Y_1$  given  $Y_2$  is:

$$\frac{f(y_1, y_2)}{f_2(y_2)} = \frac{2}{2(1-y_2)},$$

or

$$f(y_1 | y_2) = \frac{1}{1-y_2} \quad 0 \leq y_1 \leq 1 - y_2.$$

If  $Y_2 = \frac{1}{4}$  then,

$$f(y_1 | y_2 = \frac{1}{4}) = \frac{1}{\left(\frac{3}{4}\right)} = \frac{4}{3} \quad \text{for } 0 \leq y_1 \leq \frac{3}{4}$$

and hence,

$$P(Y_1 \leq \frac{1}{2} | Y_2 = \frac{1}{4}) = \int_{1/2}^{3/4} \frac{4}{3} dy_1 = \frac{4}{3} \left(\frac{1}{4}\right) = \frac{1}{3}.$$

Again notice we could have this probability without integration as the conditional density is constant.

5.30 a.  $f_1(y_1) = \int_0^1 (y_1 + y_2) dy_2 = y_1 + \frac{1}{2} \quad 0 \leq y_1 \leq 1$

$$f_2(y_2) = \int_0^1 (y_1 + y_2) dy_1 = y_2 + \frac{1}{2} \quad 0 \leq y_2 \leq 1$$

b. Calculate

$$P(Y_2 \geq \frac{1}{2}) = \int_{1/2}^1 (y_2 + \frac{1}{2}) dy_2 = \left[ \frac{1}{2} y_2 + \frac{y_2^2}{2} \right]_{1/2}^1 = \frac{5}{8}$$

$$P(Y_1 \geq \frac{1}{2}, Y_2 \geq \frac{1}{2}) = \int_{1/2}^1 \int_{1/2}^1 (y_1 + y_2) dy_1 dy_2 = \int_{1/2}^1 \left( \frac{3}{8} + \frac{y_2}{2} \right) dy_2 = \frac{3}{8}$$

Hence

$$P(Y_1 \geq \frac{1}{2} | Y_2 \geq \frac{1}{2}) = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5}$$

c. First consider  $f(y_1|y_2) = \frac{f(y_1, y_2)}{f(y_2)}$ . If  $0 \leq y_2 \leq 1$  we have

$$f(y_1|y_2) = \frac{y_1 + y_2}{y_2 + \frac{1}{2}} \quad 0 \leq y_1 \leq 1$$

Then

$$\begin{aligned} P(Y_1 > .75 | Y_2 = .5) &= \int_{.75}^1 \frac{y_1 + \frac{1}{2}}{\frac{1}{2} + \frac{1}{2}} dy_1 \\ &= \left( \frac{1}{2} y_1^2 + \left( \frac{1}{2} \right) y_1 \right]_{.75}^1 \\ &= \left( \frac{1}{2} \right) + \left( \frac{1}{2} \right) - .28125 - .375 \\ &= .34375 \end{aligned}$$

5.32 It is given that  $f_1(y_1) = 1$  for  $0 \leq y_1 \leq 1$ , where  $Y_1$  = amount stocked. Further, for a fixed value of  $Y_1$ , say  $Y_1 = y_1$ ,  $f(y_2|y_1) = \frac{1}{y_1}$  for  $0 \leq y_2 \leq y_1$ , where  $Y_2$  = amount sold.

a. By definition,

$$f(y_1, y_2) = f_1(y_1) f(y_2|y_1) = \begin{cases} \frac{1}{y_1}, & \text{for } 0 \leq y_2 \leq y_1 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

b. Given that  $Y_1 = \frac{1}{2}$ , it is necessary to find  $P(Y_2 > \frac{1}{4} | Y_1 = \frac{1}{2})$ . Using the conditional density of  $Y_2$  given  $Y_1 = \frac{1}{2}$ , which is

$$f(y_2|y_1 = \frac{1}{2}) = \frac{1}{(\frac{1}{2})} = \begin{cases} 2, & 0 \leq y_2 \leq \frac{1}{2} \\ 0, & \text{elsewhere} \end{cases}$$

we have

$$P(Y_2 > \frac{1}{4} | Y_1 = \frac{1}{2}) = \int_{1/4}^{1/2} 2 dy_2 = 2y_2 \Big|_{1/4}^{1/2} = \frac{1}{2}$$

c. The probability of interest is  $P(Y_1 \geq \frac{1}{2} | Y_2 = \frac{1}{4})$ . Hence it is necessary to calculate  $f(y_1|y_2)$ . Note that

$$f_2(y_2) = \int_{y_2}^1 \frac{1}{y_1} dy_1 = \ln y_1 \Big|_{y_2}^1 = -\ln y_2 \quad 0 \leq y_2 \leq 1.$$

Then if  $0 \leq y_2 \leq 1$

$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{1}{y_1(-\ln y_2)} \quad y_2 \leq y_1 \leq 1$$

or

$$f(y_1|y_2 = \frac{1}{4}) = \frac{1}{y_1 \ln 4} \quad \frac{1}{4} \leq y_1 \leq 1.$$

Finally,

$$P(Y_1 \geq \frac{1}{2} | Y_2 = \frac{1}{4}) = \int_{1/2}^1 \frac{1}{y_1 \ln 4} dy_1 = \frac{1}{\ln 4} \ln y_1 \Big|_{1/2}^1 = \frac{\ln 2}{\ln 4} = \frac{1}{2}$$

**5.40** No. Considering  $P(Y_1 = 3, Y_2 = 1)$  and  $p(Y_1 = 3)p(Y_2 = 1)$

$$p(3, 1) = \frac{1}{8} \neq \left(\frac{1}{8}\right)\left(\frac{4}{8}\right) = p_1(3)p_2(1)$$

Thus,  $Y_1$  and  $Y_2$  are not independent.

**5.42** Dependent, for example,  $P(Y_1 = 0, Y_2 = 0) \neq P(Y_1 = 0)P(Y_2 = 0)$ .

**5.48** Dependent as the range of  $y_1$  values on which  $f(y_1, y_2)$  is defined depends on  $y_2$ .

**5.50** Dependent as the range of  $y_1$  values on which  $f(y_1, y_2)$  is defined depends on  $y_2$ .  
More rigorously one could verify from the solution to exercise 5.28 that  $f(y_2|Y_1 = y_1) \neq f(y_2)$ .

**5.58** It is given that

$$p_1(y_1) = \binom{2}{y_1} (.2)^{y_1} (.8)^{2-y_1} \quad y_2 = 0, 1, 2; \quad p_2(y_2) = \binom{1}{y_2} (.3)^{y_2} (.7)^{1-y_2} \quad y_2 = 0, 1,$$

a.  $p(y_1, y_2) = p_1(y_1)p_2(y_2) = \binom{2}{y_1} (.2)^{y_1} (.3)^{y_2} (.8)^{2-y_1} (.7)^{1-y_2}$   
for  $y_1 = 0, 1, 2$  and  $y_2 = 0, 1$ .

b. Since  $Y_i$  is the number of customers in line  $i$ ,  $i = 1, 2$ , purchasing more than \$50 in groceries, the probability of interest is

$$\begin{aligned} P(Y_1 + Y_2 \leq 1) &= P(Y_1 = 0, Y_2 = 0) + P(Y_1 = 1, Y_2 = 0) + P(Y_1 = 0, Y_2 = 1) \\ &= (.8)^2(.7) + 2(.2)(.8)(.7) + (.3)(.8)^2 = .864 \end{aligned}$$