

# MATH 323

## Solutions to Assignment #2

5.62 a.  $E(Y_1) = np = 2 \left(\frac{1}{3}\right) = \frac{2}{3}$ .

b.  $V(Y_1) = np(1-p) = 2 \left(\frac{1}{3}\right) \left(\frac{2}{3}\right) = \frac{4}{9}$ .

c.  $E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = \left(\frac{2}{3}\right) - \left(\frac{2}{3}\right) = 0$ .

5.68 Refer to Exercise 5.14.

$$\begin{aligned} E(Y_1) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1 f(y_1, y_2) dy_1 dy_2 = \int_0^1 \int_0^1 y_1 (y_1 + y_2) dy_1 dy_2 \\ &= \int_0^1 \left[ \frac{y_1^3}{3} + \frac{y_1^2 y_2}{2} \right]_0^1 dy_2 = \left[ \frac{1}{3} y_2 + \frac{y_2^2}{4} \right]_0^1 = \frac{7}{12} \end{aligned}$$

By symmetry,  $E(Y_2) = \frac{7}{12}$  and  $E(30Y_1 + 25Y_2) = (30 + 25) \left(\frac{7}{12}\right) = 32.08$ .

5.70 The marginal distribution of  $Y_1$  is  $f_1(y_1) = 1$  for  $0 \leq y_1 \leq 1$ , so that  $E(Y_1) = \int_0^1 y_1 dy_1 = \frac{1}{2}$ . Using the joint distribution of  $Y_1$  and  $Y_2$ , we obtain

$$E(Y_2) = \int_0^1 \int_0^{y_1} \frac{y_2}{y_1} dy_2 dy_1 = \int_0^1 \frac{y_2^2}{2y_1} dy_1 = \frac{y_1^2}{4} \Big|_0^1 = \frac{1}{4}$$

Thus,  $E(Y_1 - Y_2) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ .

5.72 a. As we know  $Y_1$  and  $Y_2$  are geometric( $p$ ) random variables,  $E(Y_1) = E(Y_2) = 1/p$ .

Then  $E(Y_1 - Y_2) = 0$ .

b. We know  $V(Y_1) = \frac{1-p}{p^2} = E(Y_1^2) - E^2(Y_1) = E(Y_1^2) - \frac{1}{p^2}$ . Therefore,

$E(Y_1^2) = E(Y_2^2) = \frac{2-p}{p^2}$ . Note then  $E(Y_1 Y_2) = E(Y_1)E(Y_2) = \frac{1}{p^2}$ .

c.  $E((Y_1 - Y_2)^2) = E(Y_1^2) - 2E(Y_1 Y_2) + E(Y_2^2) = 2\left(\frac{2-p}{p^2} - \frac{1}{p^2}\right) = 2\left(\frac{1-p}{p^2}\right)$ .

$V(Y_1 - Y_2) = E((Y_1 - Y_2)^2) = 2\left(\frac{1-p}{p^2}\right)$ . Recall the following general fact. If  $Y_1$  and  $Y_2$  are independent and identically distributed then  $V(Y_1 - Y_2) = 2V(Y_1) = 2V(Y_2)$ .

d. Chebychev's theorem says (Theorem 3.14)

$$P(|Y_1 - Y_2 - 0| < 3\sigma) \geq 8/9.$$

Therefore  $(-3\sigma, 3\sigma)$  where  $\sigma = \sqrt{2\left(\frac{1-p}{p^2}\right)}$  will contain  $Y_1 - Y_2$  with probability at least  $\frac{8}{9}$ .

**5.76** Refer to the joint distribution of  $Y_1$  and  $Y_2$  given in the solution to Exercise 5.3.

$$E(Y_1) = \frac{4}{3} \quad (\text{from Exercise 5.45})$$

$$E(Y_2) = 1\left(\frac{45}{84}\right) + 2\left(\frac{18}{84}\right) + \frac{3}{84} = \frac{84}{84} = 1$$

$$E(Y_1 Y_2) = 1(1)\left(\frac{24}{84}\right) + 2(1)\left(\frac{12}{84}\right) + 1(2)\left(\frac{18}{84}\right) = \frac{84}{84} = 1$$

since all other products involve a zero term and hence add zero to the expectation.

$$\text{Thus } \text{Cov}(Y_1, Y_2) = 1 - \left(\frac{4}{3}\right) = -\frac{1}{3}.$$

**5.78** From Exercise 5.65,  $E(Y_1) = \frac{1}{4}$  and  $E(Y_2) = \frac{1}{2}$ .

$$E(Y_1 Y_2) = \int_0^1 \int_0^{y_1} 6y_1 y_2 (1 - y_2) dy_1 dy_2 = \int_0^1 3(y_2^3 - y_2^4) dy_2 = \frac{3}{4} - \frac{3}{5} = \frac{3}{20}$$

$$\text{Cov}(Y_1, Y_2) = \frac{3}{20} - \frac{1}{8} = \frac{1}{40}$$

Since  $\text{Cov}(Y_1, Y_2) \neq 0$ ,  $Y_1$  and  $Y_2$  are not independent.

**5.84** a.  $E(Y_1) = E(Z) = 0$ .  $E(Y_2) = E(Z^2) = V(Z) = 1$ .

b.  $E(Y_1 Y_2) = E(Z^3)$ . To find the expected value of the cube of a standard normal, we will use the method of moment generating functions. Recall the mgf of a standard normal is  $e^{t^2/2}$ . Note then

$$\frac{\partial^3}{\partial t^3} e^{t^2/2} = \frac{\partial^2}{\partial t^2} (te^{t^2/2}) = \frac{\partial}{\partial t} (e^{t^2/2} + t^2 e^{t^2/2}) = te^{t^2/2} + 2te^{t^2/2} + t^3 e^{t^2/2}$$

which is 0 when  $t = 0$ .

$$\text{c. } \text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = 0 - 0 = 0.$$

d.  $Y_1$  and  $Y_2$  are not independent as  $P(Y_2 > 1 | Y_1 > 1) = 1 \neq P(Y_2 > 1)$ . This is an example where 0 covariance does not imply independence. Do not confuse this with the result that two normally distributed random variables are independent if and only if their covariance is 0. Specifically  $Y_2$  is not normally distributed rather it is a function of a normally distributed random variable (in specific  $Y_2$  is distributed as a chi-squared with 1 degree of freedom).

**5.88** a. The probability distribution of  $X = Y_1 + Y_2$  can be found from the table given in the solution to Exercise 5.3 as follows.

$x$	1	2	3
$p(x)$	$\frac{7}{84}$	$\frac{42}{84}$	$\frac{35}{84}$

Then

$$E(X) = \frac{7}{84} + \frac{84}{84} + \frac{105}{84} = \frac{196}{84} = \frac{7}{3}$$

$$E(X^2) = \frac{7}{84} + \frac{168}{84} + \frac{315}{84} = \frac{490}{84}$$

$$V(X) = \frac{490}{84} - \frac{49}{9} = .3889$$

b. From previous exercises,  $E(Y_1) = \frac{4}{3}$ ,  $E(Y_2) = 1$ ,  $\text{Cov}(Y_1, Y_2) = -\frac{1}{3}$ . Calculate  $V(Y_1) = 1\left(\frac{40}{84}\right) + 4\left(\frac{30}{84}\right) + 9\left(\frac{4}{84}\right) - \frac{16}{9} = \frac{10}{18}$

and

$$V(Y_2) = 1\left(\frac{45}{84}\right) + 4\left(\frac{18}{84}\right) + 9\left(\frac{1}{84}\right) - 1 = \frac{42}{84}.$$

Using Theorem 5.12, we obtain

$$E(Y_1 + Y_2) = \frac{7}{3}$$

$$V(Y_1 + Y_2) = \frac{10}{18} + \frac{42}{84} + 2\left(-\frac{1}{3}\right) = \frac{7}{18} = .3889$$

**5.90**  $V(Y_1 - 3Y_2) = \frac{3}{80} + 9\left(\frac{1}{20}\right) - 6\left(\frac{1}{40}\right) = \frac{27}{80} = .3375$   
 (See Exercise 5.65 for  $V(Y_1)$  and  $V(Y_2)$ .)

- 5.99** a. Using the multinomial distribution with  $p_1 = p_2 = p_3 = \frac{1}{3}$ ,
- $$P(Y_1 = 3, Y_2 = 1, Y_3 = 2) = \frac{6!}{3!1!2!} = \left(\frac{1}{3}\right)^3 \left(\frac{1}{3}\right)^1 \left(\frac{1}{3}\right)^2 = 60 \left(\frac{1}{3}\right)^6 = .0823$$
- b. Refer to Theorem 5.13 in the text.
- $$E(Y_1) = np_1 = \frac{n}{3} \quad \text{and} \quad V(Y_1) = np_1q_1 = \frac{2n}{9}$$
- c. Refer to Theorem 5.13 in the text.
- $$\text{Cov}(Y_1, Y_2) = -np_2p_3 = -n\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = -\frac{n}{9}$$
- d.  $E(Y_2 - Y_3) = \frac{n}{3} - \frac{n}{3} = 0 \quad \text{and}$   

$$V(Y_2 - Y_3) = \frac{2n}{9} + \frac{2n}{9} - 2\left(-\frac{n}{9}\right) = \frac{6n}{9} = \frac{2n}{3}$$

**5.100**  $E(C) = E(Y_1) + 3E(Y_2) = np_1 + 3np_2$   
 $V(C) = V(Y_1) + 9V(Y_2) + 6\text{Cov}(Y_1, Y_2) = np_1q_1 + 9np_2q_2 - 6np_1p_2$

**5.106**  $E(Y_1) = 10(.01) = .1; V(Y_1) = 10(.1)(.9) = .9.$   
 $E(Y_2) = 10(.05) = .5; V(Y_2) = 10(.05)(.95) = .475.$   
 $\text{Cov}(Y_1, Y_2) = -10(.1)(.05) = -.05;$   
 $E[Y_1 + 3Y_2] = E(Y_1) + 3E(Y_2) = 1 + 3(.5) = 2.5.$   
 $V[Y_1 + 3Y_2] = V(Y_1) + 3^2V(Y_2) + 2(3)\text{Cov}(Y_1, Y_2) = .9 + 9(.475) + 6(-.05) = 4.875.$

## 6.2 Calculate

$$F_Y(y) = \int_{-1}^y \frac{3}{2} t^2 dt = \frac{1}{2} (y^3 + 1)$$

for  $-1 \leq y \leq 1$

a.  $F_{U_1}(u) = P(3Y \leq u) = P\left(Y \leq \frac{u}{3}\right) = F_Y\left(\frac{u}{3}\right) = \frac{1}{2} \left(\frac{u^3}{27} + 1\right) \quad \text{for } -3 \leq u \leq 3$

Differentiating with respect to  $u$ , we have

$$f_{U_1}(u) = \begin{cases} \frac{u^2}{18}, & -3 \leq u \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

b.  $F_{U_2}(u) = P(3 - Y \leq u) = P(Y \geq 3 - u) = 1 - F_Y(3 - u) = \frac{1}{2} [1 - (3 - u)^3]$   
 for  $-1 \leq (3 - u) \leq 1$  or  $2 \leq u \leq 4$ . Differentiating with respect to  $u$ , we have

$$f_{U_2}(u) = \begin{cases} \left(\frac{3}{2}\right)(3 - u)^2, & 2 \leq u \leq 4 \\ 0, & \text{elsewhere} \end{cases}$$

c.  $F_{U_3}(u) = P(Y^2 \leq u) = P(-\sqrt{u} \leq Y \leq \sqrt{u}) = F_Y(\sqrt{u}) - F_Y(-\sqrt{u})$   
 $= \frac{1}{2} u^{3/2} + \frac{1}{2} u^{3/2} = u^{3/2}$  for  $0 \leq u^2 \leq 1$  or  $0 \leq u \leq 1$ .

Hence

$$f_{U_3}(u) = \frac{3}{2} u^{1/2} = \frac{3}{2} \sqrt{u}$$

for  $0 \leq u \leq 1$

6.4 It is given that  $f(y) = \left(\frac{1}{4}\right) e^{-y/4}$  for  $y \geq 0$ . Then

$$F_Y(y) = \int_0^y \frac{1}{4} e^{-t/4} dt = 1 - e^{-y/4}, \quad y > 0$$

a.  $F_U(u) = P(3Y + 1 \leq u) = P(Y \leq \frac{u-1}{3}) = 1 - e^{-(u-1)/12}$  for  $u \geq 1$

Finally,

$$f_U(u) = \frac{d}{du} F_U(u) = \begin{cases} \left(\frac{1}{12}\right) e^{-(u-1)/12}, & u \geq 1 \\ 0, & \text{elsewhere} \end{cases}$$

b.  $E(U) = \int_1^\infty \frac{u}{12} e^{-(u-1)/12} du = e^{12} \int_1^\infty \frac{u}{12} e^{-u/12} = 12e^{12} \int_{1/12}^\infty z e^{-z} dz = 13$

where  $z = u/12$  and we used integration by parts to evaluate the integral.

6.8 The region for which  $Y_1 - Y_2 \leq u$  is shown in Figure 6.3 over the region for which  $Y_1$  and  $Y_2$  are positive and  $y_2 \leq y_1$ .

a.  $F_U(u) = P(Y_1 - Y_2 \leq u)$

$$= \int_0^\infty \int_{y_2}^{y_1+u} e^{-y_1} dy_1 dy_2$$

for  $u \geq 0$ . Then  $f_U(u) = e^{-u}$  for  $u \geq 0$ , which is a gamma density with  $\alpha = \beta = 1$ .

b.  $E(U) = \alpha\beta = 1; V(U) = \alpha\beta^2 = 1$ .

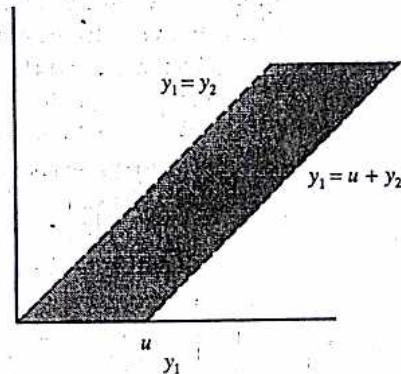


Figure 6.3

6.22 a. Let  $U = Y^m$ . Then using the transformation approach, we have

$$Y = U^{1/m}$$

and

$$\frac{dy}{du} = \frac{1}{m} u^{(1-m)/m} = \frac{1}{m} u^{-(m-1)/m}$$

so that

$$g_U(u) = \frac{1}{\alpha} m (u^{1/m})^{m-1} e^{-u/\alpha} \times \frac{1}{m} u^{-(m-1)/m} = \frac{1}{\alpha} e^{-u/\alpha}$$

for  $u > 0$ .

b.  $E(Y^k) = E(U^{k/m}) = \int_0^\infty \frac{u^{k/m} e^{-u/\alpha}}{\alpha} du = \frac{1}{\alpha} \Gamma\left(\frac{k}{m} + 1\right) \alpha^{(k/m)+1} = \Gamma\left(\frac{k}{m} + 1\right) \alpha^{k/m}$

Note that the integrand is the density (except for constants) of a gamma variable with parameters  $(\frac{k}{m}) + 1$  and  $\alpha$ , so that integration can be done by choosing the necessary constants.

6.24  $f(y) = 1 \quad \text{for } 0 \leq y \leq 1$

$U = -2 \ln Y$ ; solving for  $y$ ,

$$\ln y = -\frac{u}{2} \text{ which gives } y = e^{-u/2}.$$

$$\frac{dy}{du} = \left(-\frac{1}{2}\right) e^{-u/2}.$$

Thus,

$$f_u(u) = f(y) \left| \frac{dy}{du} \right| = 1 \left| \left(-\frac{1}{2}\right) e^{-u/2} \right| = \left(\frac{1}{2}\right) e^{-u/2} \quad \text{for } u > 0$$

which is exponential with  $\beta = 2$ .

6.26 It is given that  $f(i) = \frac{1}{2}$ , for  $9 \leq i \leq 11$ . If  $P = 2i^2$ , then  $i = \sqrt{\frac{P}{2}}$  and

$$\left| \frac{di}{dp} \right| = \left(\frac{1}{4}\right) \left(\frac{p}{2}\right)^{-1/2} = \left(\frac{1}{2}\right)^{3/2} p^{-1/2}.$$

Then

$$f_p(p) = \frac{1}{2^{5/2} p^{1/2}} = \frac{1}{4\sqrt{2p}}$$

for  $9 \leq \sqrt{\frac{p}{2}} \leq 11$  or in other words  $162 \leq p \leq 242$ .

6.28 If  $U = 2Y^2 + 3$ , then  $Y = \left(\frac{U-3}{2}\right)^{1/2}$  and  $\left| \frac{dy}{du} \right| = \left(\frac{1}{4}\right) \left(\frac{\sqrt{2}}{\sqrt{u-3}}\right)$ . Then

$$f_U(u) = \frac{\sqrt{2}}{16\sqrt{u-3}} = \frac{1}{8\sqrt{2(u-3)}} \quad \text{for } 1 \leq \left(\frac{u-3}{2}\right)^{1/2} \leq 5 \quad \text{or} \quad 5 \leq u \leq 53$$

$$f_U(u) = 0 \quad \text{elsewhere}$$

6.54 a. First note that, by independence, we have

$$f_{Y_1 Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} y_1^{\alpha_1-1} y_2^{\alpha_2-1} e^{-(y_1+y_2)/\beta}.$$

Now let  $U = \frac{Y_1}{Y_1+Y_2}$  and  $V = Y_1 + Y_2$ . Then  $y_1 = uv$  and  $y_2 = v(1-u)$  so that the Jacobian of the transformation is  $J = v$  (see example 6.14). Thus, the joint density of  $U$  and  $V$  is

$$\begin{aligned} f_{U,V}(u, v) &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} (uv)^{\alpha_1-1} \{v(1-u)\}^{\alpha_2-1} e^{-v/\beta} v \\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} u^{\alpha_1-1} (1-u)^{\alpha_2-1} v^{\alpha_1+\alpha_2-1} e^{-v/\beta} \end{aligned}$$

when  $0 < u < 1$  and  $v > 0$ .

(continued)

## (6.54) (continued)

b. Notice that

$$\begin{aligned} f_U(u) &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} u^{\alpha_1-1}(1-u)^{\alpha_2-1} \int_0^\infty v^{\alpha_1+\alpha_2-1} e^{-v/\beta} dv \\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} u^{\alpha_1-1}(1-u)^{\alpha_2-1} \Gamma(\alpha_1 + \alpha_2) \beta^{\alpha_1+\alpha_2} \\ &= \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1-1}(1-u)^{\alpha_2-1} \end{aligned}$$

for  $0 < u < 1$ . Thus,  $U \sim \text{Beta}(\alpha_1, \alpha_2)$ .

- c. This problem is very straightforward using the method of moment generating functions.  
Alternatively we could calculate the density of  $V$  directly,

$$\begin{aligned} f_V(v) &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1} v^{\alpha_1+\alpha_2-1} e^{-v/\beta} du \\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} v^{\alpha_1+\alpha_2-1} e^{-v/\beta} \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)} \\ &= \frac{1}{\Gamma(\alpha_1 + \alpha_2)\beta^{\alpha_1+\alpha_2}} v^{\alpha_1+\alpha_2-1} e^{-v/\beta} \end{aligned}$$

for  $v > 0$ . Thus,  $V \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$ .

- d.  $f_{U,V}(u, v) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} u^{\alpha_1-1}(1-u)^{\alpha_2-1} v^{\alpha_1+\alpha_2-1} e^{-v/\beta}$   
 $= \left( \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1-1}(1-u)^{\alpha_2-1} \right) \left( \frac{1}{\Gamma(\alpha_1 + \alpha_2)} v^{\alpha_1+\alpha_2-1} e^{-v/\beta} \right)$   
 $= f_U(u)f_V(v)$  which implies the independence of  $U$  and  $V$ .

## 6.56 a. By independence we have

$$f_{Y_1, Y_2}(y_1, y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2) = 1$$

for  $0 < y_1 < 1$   $0 < y_2 < 1$ . Then the inverse transformations are

$$y_1 = \frac{u+v}{2}$$

and

$$y_2 = \frac{u-v}{2}.$$

Thus the Jacobian is

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

so that

$$f_{U,V}(u, v) = \frac{1}{2}$$

for  $0 < \frac{u+v}{2} < 1$ , and  $0 < \frac{u-v}{2} < 1$ . or, taking on a case by case basis, the support may be expressed as  $0 < u < 1$ ,  $-u < v < u$  or  $1 \leq u \leq 2$ ,  $u-2 < v < 2-u$ .

- c. If  $0 < u < 1$  then

$$f_U(u) = \int_{-u}^1 \frac{1}{2} dv = u$$

but if  $1 \leq u < 2$  then

$$f_U(u) = \int_{u-2}^{2-u} \frac{1}{2} dv = 2-u.$$

- d. If  $-1 < v < 0$  then

$$f_V(v) = \int_{-v}^{2+v} \frac{1}{2} du = 1+v$$

but if  $1 \leq u < 2$  then

$$f_V(v) = \int_v^{2-v} \frac{1}{2} du = 1-u.$$

- e. No, as their joint density does not factor. Specifically the range of values  $U$  can take depends on  $V$  and vice versa.