

MATH 323

Solutions to Assignment No. 3

6.34 Since Y_1 and Y_2 are independent standard normal random variables, the moment-generating functions for Y_1^2 and Y_2^2 can be written, from Example 6.11, as

$$m_{Y_1^2}(t) = \frac{1}{(1-2t)^{1/2}}$$

$$m_{Y_2^2}(t) = \frac{1}{(1-2t)^{1/2}}.$$

Now using Theorem 6.2, we have

$$m_U(t) = m_{Y_1^2}(t) m_{Y_2^2}(t) = \frac{1}{(1-2t)^1},$$

which is the moment-generating function of a gamma random variable with $\alpha = 1$ and $\beta = 2$. Hence by Theorem 6.1, U has a gamma distribution with $\alpha = 1$ and $\beta = 2$.

Equivalently, U has a χ^2 distribution with 2 degrees of freedom.

$$\begin{aligned} \text{6.40 } m_W(t) &= E(e^{tW}) = E(e^{t2Y/\beta}) = m_Y\left(\frac{2t}{\beta}\right) = E(e^{(2t/\beta)Y}) \\ &= \left[\frac{1}{1-\beta\left(\frac{2t}{\beta}\right)}\right]^\pi = \frac{1}{(1-2t)^{n/2}}. \end{aligned}$$

Thus W has a gamma distribution with $\alpha = \frac{n}{2}$ and $\beta = 2$, i.e., a χ^2 distribution with n degrees of freedom.

6.44 a. Because Y_1 and Y_2 are Poisson random variables,

$$m_{Y_1}(t) = e^{\lambda_1(e^t-1)}$$

and

$$m_{Y_2}(t) = e^{\lambda_2(e^t-1)}.$$

So that $m_{Y_1+Y_2}(t) = \exp[-(\lambda_1 + \lambda_2)(1 - e^t)]$; which is the moment-generating function of a Poisson random variable with mean $\lambda_1 + \lambda_2$. (Recall that $\exp()$ is simply a convenient way to write $e^()$). By Theorem 6.1, then,

$$P(Y_1 + Y_2 = k) = \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^k}{k!}$$

for $k = 0, 1, 2, \dots$

b. By definition,

$$P(Y_1 = k | Y_1 + Y_2 = m) = \frac{P(Y_1=k, Y_1+Y_2=m)}{P(Y_1+Y_2=m)} = \frac{P(Y_1=k, Y_2=m-k)}{P(Y_1+Y_2=m)}.$$

$$= \frac{\left[\frac{e^{-\lambda_1} \lambda_1^k}{k!} \right] \left[\frac{e^{-\lambda_2} \lambda_2^{m-k}}{(m-k)!} \right]}{\left[e^{-(\lambda_1+\lambda_2)} \left(\frac{(\lambda_1+\lambda_2)^m}{m!} \right) \right]} = \binom{m}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2} \right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2} \right)^{m-k}, \quad k = 0, 1, 2, \dots, m$$

which is the probability distribution function for a binomial random variable with parameters m and $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

6.62 a. The density and cdf of the Y_i are

$$f(y) = \begin{cases} \frac{1}{\theta}, & 0 \leq y \leq \theta \\ 0, & \text{otherwise} \end{cases}$$

and

$$F(y) = \begin{cases} 0, & y < 0 \\ \frac{y}{\theta}, & 0 \leq y \leq \theta \\ 1, & y > \theta \end{cases}$$

Then, from Theorem 6.5,

$$f_{(k)}(y_k) = \frac{n!}{(k-1)!(n-k)!} \left(\frac{y}{\theta}\right)^{k-1} \left[\frac{\theta-y}{\theta}\right]^{n-k} \left(\frac{1}{\theta}\right)$$

$$= \frac{n!}{(k-1)!(n-k)!} \frac{y^{k-1}(\theta-y)^{n-k}}{\theta^n} \quad 0 \leq y_k \leq \theta.$$

$$\text{b. } E(Y_{(k)}) = \int_0^\theta y f_{(k)}(y) dy = \int_0^\theta \frac{n!}{(k-1)!(n-k)!} \frac{y^k(\theta-y)^{n-k}}{\theta^n} dy$$

$$= \frac{k}{n+1} \int_0^\theta \frac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(n-k+1)} \left(\frac{y}{\theta}\right)^k \left(1 - \frac{y}{\theta}\right)^{n-k} dy$$

Let $z = \frac{y}{\theta}$ and $dy = \theta dz$. Then

$$E(Y_{(k)}) = \frac{k}{n+1} \theta \int_0^1 \frac{z^k(1-z)^{n-k}}{B(k+1, n-k+1)} dz$$

Notice that the integral is that y is beta density with $\alpha = k+1$ and $\beta = n-k+1$ and, hence, the integral is 1. Then

$$E(Y_{(k)}) = \frac{k}{n+1} \theta$$

c. Similar to part b.

$$E(Y_{(k)}^2) = \int_0^\theta y^2 \frac{n!}{(k-1)!(n-k)!} \left(\frac{y}{\theta}\right)^{k-1} \left(1 - \frac{y}{\theta}\right)^{n-k} \left(\frac{1}{\theta}\right) dy$$

Letting $z = y/\theta$, this becomes

$$\frac{\theta^2 k(k+1)}{(n+1)(n+2)} \int_0^1 \frac{\Gamma(n+3)}{\Gamma(k+2)\Gamma(n-k+1)} z^{k+1} (1-z)^{n-k} dz = \frac{\theta^2 k(k+1)}{(n+1)(n+2)}.$$

$$\text{Then } V(Y_{(k)}) = E(Y_{(k)}^2) - [E(Y_{(k)})]^2$$

$$= \frac{\theta^2 k(k+1)}{(n+1)(n+2)} - \frac{k^2 \theta^2}{(n+1)^2} = \frac{\theta^2 k}{n+1} \left[\frac{k+1}{n+2} - \frac{k}{n+1} \right] = \frac{(n-k+1)k\theta^2}{(n+1)^2(n+2)}$$

$$\text{d. } E(Y_{(k)} - Y_{(k-1)}) = E(Y_{(k)}) - E(Y_{(k-1)}) = \frac{k\theta}{n+1} - \frac{(k-1)\theta}{n+1} \quad (\text{by part b.})$$

$$= \frac{\theta}{n+1},$$

which is constant for all k . Thus, the order statistics are, on the average, equally spaced.

$$6.64 f_Y(y) = \left[\frac{1}{B(2, 2)} \right] y(1-y) = \left[\frac{\Gamma(4)}{\Gamma(2)\Gamma(2)} \right] y(1-y) = 6y(1-y) \quad 0 \leq y \leq 1.$$

$$F_Y(y) = \int_0^y 6y(1-y) dy = 3y^2 - 2y^3 \quad 0 \leq y \leq 1$$

$$\text{a. } [F_Y(y)]^n = (3y^2 - 2y^3)^n$$

Thus,

$$F_{Y_n}(y) = \begin{cases} 0, & y < 0 \\ (3y^2 - 2y^3)^n, & 0 \leq y \leq 1 \\ 1, & y > 1 \end{cases}$$

$$\text{b. } f_{Y_n}(y) = n(3y^2 - 2y^3)^{n-1} (6y - 6y^2) = 6ny(1-y)(3y^2 - 2y^3)^{n-1}, \quad 0 \leq y \leq 1$$

c. For $n = 2$,

$$f_{Y_2}(y) = 12y(1-y)(3y^2 - 2y^3) = 36y^3 - 60y^4 + 24y^5, \quad 0 \leq y \leq 1.$$

$$E(Y_{(2)}) = \int_0^1 y(36y^3 - 60y^4 + 24y^5) dy = \frac{36}{5} y^5 \Big|_0^1 - \frac{60}{6} y^6 \Big|_0^1 + \frac{24}{7} y^7 \Big|_0^1 = .6286$$

- 6.66 a.** For the exponential random variable, the density function is

$$f(y) = \frac{1}{\beta} e^{-y/\beta}$$

for $0 \leq y < \infty$ while the cumulative distribution function is given by

$$F(y) = 1 - e^{-y/\beta}$$

for $0 \leq y < \infty$. Then, from Theorem 6.5,

$$\begin{aligned} f_{(k)}(y_k) &= \frac{n!}{(k-1)!(n-k)!} (1 - e^{-y_k/\beta})^{k-1} (e^{-y_k/\beta})^{n-k} \left(\frac{1}{\beta} e^{-y_k/\beta}\right), \quad 0 < y_k < \infty \\ &= \frac{n!}{(k-1)!(n-k)!} \left(\frac{1}{\beta}\right) (e^{-y_k/\beta})^{n-k+1} (1 - e^{-y_k/\beta})^{k-1} \end{aligned}$$

$$\begin{aligned} \text{b. } f_{(j)(k)}(y_j, y_k) &= \frac{n!}{(j-1)!(k-1-j)!(n-k)!} \times F^{j-1}(y_j) [F(y_k) - F(y_j)]^{k-1-j} \\ &\quad \times [1 - F(y_k)]^{n-k} f(y_j) f(y_k) \quad -\infty < y_j < y_k < \infty \\ &= \frac{n!}{(j-1)!(k-1-j)!(n-j)!} [1 - e^{-y_j/\beta}]^{j-1} [e^{-y_j/\beta} - e^{-y_k/\beta}]^{k-1-j} \\ &\quad \times [e^{-y_k/\beta}]^{n-k} \left(\frac{1}{\beta} e^{-y_j/\beta}\right) \left(\frac{1}{\beta} e^{-y_k/\beta}\right) \\ &= \frac{1}{\beta^2} \frac{n!}{(j-1)!(k-j-1)!(n-j)!} [1 - e^{-y_j/\beta}]^{j-1} [e^{-y_j/\beta}] \\ &\quad \times [e^{-y_j/\beta} - e^{-y_k/\beta}]^{k-j-1} [e^{-y_k/\beta}]^{n-k+1} \end{aligned}$$

- 6.68 a.** This is similar to Exercise 6.67. Calculate

$$1 - F(y) = \int_y^\infty e^{-(t-\theta)} dt = e^{-(y-\theta)}$$

Then

$$g_1(y) = n [e^{-(y-\theta)}]^{n-1} e^{-(y-\theta)} = n e^{-n(y-\theta)}$$

for $y \geq 0$.

$$\begin{aligned} \text{b. } E(Y_{(1)}) &= \int_0^\infty n y e^{-n(y-\theta)} dy = n \int_0^\infty (z + \theta) e^{-nz} dz \quad (\text{with } z = y - \theta) \\ &= n \Gamma(2) \left(\frac{1}{n}\right)^2 + n \theta \left(\frac{1}{n}\right)^1 = \frac{1}{n} + \theta \end{aligned}$$

Notice that the integral is calculated in two parts, using the fact that the constants associated with the integrals must be constants of gamma random variables.

- 6.74** Y_1 and Y_2 are both independently and identically distributed as gamma random variables with parameters $\alpha = 2$ and $\beta = 2$. Hence the moment-generating function for Y_i , $i = 1, 2$, is

$$m_{Y_i}(t) = (1 - \beta t)^{-\alpha} = (1 - 2t)^{-2}$$

Now

$$m_{Y_{1/2}}(t) = m_{Y_1}\left(\frac{t}{2}\right) = (1 - t)^{-2}$$

and

$$m_U(t) = m_{Y_{1/2}}(t)m_{Y_{1/2}}(t) = (1 - t)^{-4}$$

Evidently U has a gamma distribution with $\alpha = 4$ and $\beta = 1$.

6.80 Let $U = Y_1 - Y_2$. We want to find $P(U = u) = P(Y_1 - Y_2 = u)$. If $u > 0$ then $y_1 = u + y_2 > 0$ so that we can write

$$\begin{aligned} P(U = u) &= P(Y_1 - Y_2 = u) \\ &= \sum_{y_2=1}^{\infty} P(Y_1 = u + y_2) P(Y_2 = y_2) \\ &= p^2(1-p)^u \sum_{y_2=1}^{\infty} (1-p)^{2y_2-1} \\ &= \frac{p^2(1-p)^{u+1}}{(1-(1-p)^2)} \sum_{y_2=1}^{\infty} [1-(1-p)^2][(1-p)^2]^{y_2-1} \\ &= \frac{p^2(1-p)^{u+1}}{(1-(1-p)^2)} \\ &= \frac{p(1-p)^{u+1}}{(2-p)} \end{aligned}$$

Where the infinite sum is 1 as it is the sum of values of a geometric random variable with success probability $1 - (1-p)^2$. On the other hand, if $u < 0$ then $y_2 = y_1 - u > 0$ and we can proceed in a fashion similar to the above approach. Specifically,

$$\begin{aligned} P(U = u) &= P(Y_1 - Y_2 = u) = \sum_{y_1=1}^{\infty} P(Y_2 = y_1 - u) P(Y_1 = y_1) \\ &= p^2(1-p)^{-u-1} \sum_{y_1=1}^{\infty} (1-p)^{2y_1} = \frac{p^2(1-p)^{-u+1}}{(1-(1-p)^2)} = \frac{p(1-p)^{-u+1}}{(2-p)}. \end{aligned}$$

Putting these results together gives us

$$P(u = u) = \frac{p(1-p)^{|u|+1}}{(2-p)}.$$

6.90 The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$ or $r = (\frac{3}{4\pi}V)^{1/3}$ with

$$\left| \frac{dr}{dV} \right| = \left(\frac{1}{3} \right) \left(\frac{3}{4\pi} \right)^{1/3} V^{-2/3}. \text{ Making the transformation, we have}$$

$$f(v) = 2 \left(\frac{3V}{4\pi} \right)^{1/3} \frac{1}{3} \left(\frac{3}{4\pi} \right)^{1/3} V^{-2/3} = \frac{2}{3} \left(\frac{3}{4\pi} \right)^{2/3} V^{-1/3} \text{ for } 0 \leq V \leq \frac{4\pi}{3}$$

- 7.10** a. Since a χ^2 random variable is defined as a gamma random variable with $\alpha = \frac{\nu}{2}$ and $\beta = 2$, the expected value and variance are

$$E(U) = \alpha\beta = \nu \quad \text{and} \quad V(U) = \alpha\beta^2 = 2\nu$$

- b. Using Theorem 7.3, we see that the quantity $\frac{(n-1)S^2}{\sigma^2}$ has a χ^2 distribution with $\nu = n - 1$. Hence, from part a,

$$E\left(\frac{(n-1)S^2}{\sigma^2}\right) = n - 1 \quad \text{or} \quad \left(\frac{n-1}{\sigma^2}\right) E(S^2) = n - 1 \quad \text{or} \quad E(S^2) = \sigma^2$$

Similarly,

$$V\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n - 1) \quad \text{or} \quad \left[\frac{(n-1)^2}{\sigma^4}\right] V(S^2) = 2(n - 1) \quad \text{or} \quad V(S^2) = \frac{2\sigma^4}{n-1}$$

7.70 For the given random variable Y ,

$$\mu = E(Y) = \int_0^1 3y^3 dy = \frac{3}{4}$$

and

$$\sigma^2 = V(Y) = \int_0^1 3y^4 dy - \frac{9}{16} = \frac{3}{5} - \frac{9}{16} = .0375$$

Using the Central Limit Theorem,

$$P(\bar{Y} > .7) = P\left(Z > \frac{.7 - .75}{\sqrt{\frac{.0375}{40}}}\right) = P(Z > -1.63) = P(Z < 1.63) = 1 - .0516 = .9484$$

7.71 a. Note that $E(X_i) = 1$ and $V(X_i) = 2$. By the Central Limit Theorem,

$$\bar{X} \sim N\left(1, \sqrt{\frac{2}{n}}\right)$$

or

$$\frac{\bar{Y} - 1}{\sqrt{\frac{2}{n}}} = \frac{\bar{Y} - n}{\sqrt{2n}} \sim N(0, 1).$$

- b. It is given that Y has a normal distribution with mean $\mu = 6$ and variance $\sigma^2 = .2$. Let C_i be the cost for a single rod, $i = 1, 2, \dots, 50$. That is, $C_i = 4(Y_i - \mu)^2$. The total cost for the day, then, will be

$$\sum_{i=1}^{50} C_i = 4 \sum_{i=1}^{50} (y_i - \mu)^2$$

where the Y_i are independent and distributed as Y above. Recall from Chapter 6 that

$$\frac{\sum_{i=1}^{50} (Y_i - \mu)^2}{\sigma^2} = \sum_{i=1}^{50} \left(\frac{Y_i - \mu}{\sigma}\right)^2 = \sum_{i=1}^{50} Z_i^2$$

has a chi-square distribution with 50 degrees of freedom, since each Z_i is standard normal, and hence Z_i^2 has a chi-square distribution with 1 degree of freedom, $i = 1, 2, \dots, 50$. The probability of interest, using the results of part a, is

$$P(\sum C_i > 48) = P[\sum (Y_i - \mu)^2 > 12] = P\left[\frac{\sum (Y_i - \mu)^2}{.2} > 60\right] = P(X > 60)$$

where X is a chi-square random variable with $n = 50$ degrees of freedom. Hence the approximation is

$$P(X > 60) = P\left(\frac{\bar{X} - n}{\sqrt{2n}} > \frac{60 - 50}{\sqrt{100}}\right) = P(Z > 1) = .1587$$

8.2

a. $E[\hat{\theta}_3] = E[a\hat{\theta}_1 + (1-a)\hat{\theta}_2] = aE[\hat{\theta}_1] + (1-a)E[\hat{\theta}_2] = a\theta + (1-a)\theta = \theta$

- b. It is given that $E(\hat{\theta}_1) = E(\hat{\theta}_2) = \theta$ and $V(\hat{\theta}_1) = \sigma_1^2$, $V(\hat{\theta}_2) = \sigma_2^2$. Assuming that $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent, the variance of the new estimator, $\hat{\theta}_3$, will be

$$V(\hat{\theta}_3) = V[a\hat{\theta}_1 + (1-a)\hat{\theta}_2] = a^2 V(\hat{\theta}_1) + (1-a)^2 V(\hat{\theta}_2) = a^2 \sigma_1^2 + (1-a)^2 \sigma_2^2$$

In order to choose a value of a such that $V(\hat{\theta}_3)$ is minimized, look at

$$\frac{d}{da} V(\hat{\theta}_3) = 2a\sigma_1^2 - 2(1-a)\sigma_2^2.$$

Setting the derivative equal to 0, we obtain

$$a\sigma_1^2 - (1-a)\sigma_2^2 = 0$$

or

$$a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

Notice that $\frac{d^2}{da^2} V(\hat{\theta}_3) = 2\sigma_1^2 + 2\sigma_2^2 > 0$, so that the value is in fact a minimum.

- 8.4** Recall that if Y_i is Exponential(θ) then $E(Y_i) = \theta$ and $V(Y_i) = \theta^2$. Hence we can use Theorem 5.12 to obtain

$$\begin{aligned} E(\hat{\theta}_1) &= E(\hat{\theta}_2) = E(\hat{\theta}_3) = E(\hat{\theta}_5) = \theta \\ V(\hat{\theta}_1) &= \theta^2 \\ V(\hat{\theta}_2) &= \frac{1}{4}(2\theta^2) = \frac{\theta^2}{2} \\ V(\hat{\theta}_3) &= \frac{1}{9}(\theta^2 + 4\theta^2) = \frac{5\theta^2}{9} \\ V(\hat{\theta}_5) &= \frac{1}{9}(3\theta^2) = \frac{\theta^2}{3} \end{aligned}$$

The distribution of $\hat{\theta}_4$ can be obtained by using the methods of Section 6.6 in the text, with $F(y) = 1 - e^{-y/\theta}$. Then

$$g_1(y) = \frac{3}{\theta} e^{-y/\theta} (e^{-y/\theta})^2 = \frac{3}{\theta} e^{-3y/\theta}$$

which is an exponential distribution with mean $\frac{\theta}{3}$.

$$E(\hat{\theta}_4) = \frac{\theta}{3} \quad V(\hat{\theta}_4) = \frac{\theta^2}{9}$$

- a. The unbiased estimators are $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3$, and $\hat{\theta}_5$.
- b. Among these four estimators, $\hat{\theta}_5 = \bar{Y}$ has the smallest variance.

- 8.6**
- a. For the Poisson distribution, $E(Y_i) = \lambda$ and $E(\bar{Y}) = \lambda$. Hence $\hat{\lambda} = \bar{Y}$ is an unbiased estimator for λ .
 - b. In order to find $E(Y^2)$, use the fact that $V(Y) = \lambda$ and $E(Y^2) = V(Y) + [E(Y)]^2 = \lambda + \lambda^2$. Then $E(C) = 3E(Y) + E(Y^2) = 4\lambda + \lambda^2$.
 - c. Since $E(\bar{Y}) = \lambda$, $E(\bar{Y})^2 = V(\bar{Y}) + [E(\bar{Y})]^2 = \frac{\lambda}{n} + \lambda^2$, we construct as an estimator $\hat{\theta} = \bar{Y}^2 + \bar{Y}\left(4 - \frac{1}{n}\right)$. Considering

$$E(\hat{\theta}) = \frac{\lambda}{n} + \lambda^2 + 4\lambda - \left(\frac{1}{n}\right)\lambda = 4\lambda + \lambda^2.$$

Thus, $\hat{\theta}$ is an unbiased estimator of $E(C)$.

- 8.8**
- a. For the uniform distribution given here, $E(Y_i) = \theta + \frac{1}{2}$. Hence $E(\bar{Y}) = \theta + \frac{1}{2}$ and the bias is $B = E(\bar{Y}) - \theta = \frac{1}{2}$.
 - b. An unbiased estimator of θ can be constructed by using $\hat{\theta} = \bar{Y} - \frac{1}{2}$, which has $E(\hat{\theta}) = \theta$.
 - c. If \bar{Y} is used as an estimator, then $V(\bar{Y}) = \frac{V(Y)}{n} = \frac{1}{12n}$ and $MSE = V(\bar{Y}) + B^2 = \frac{1}{12n} + \frac{1}{4}$.

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8.10 The following information is required to answer the question.

$$E(Y) = \int_0^\theta \left[\frac{\alpha y^n}{\theta^n} \right] dy = \left[\frac{\alpha y^{n+1}}{(n+1)\theta^n} \right]_0^\theta = \frac{\alpha \theta}{n+1}$$

$$E(Y^2) = \int_0^\theta \left[\frac{\alpha y^{n+1}}{\theta^n} \right] dy = \left[\frac{\alpha y^{n+2}}{(n+2)\theta^n} \right]_0^\theta = \frac{\alpha \theta^2}{n+2}$$

$$f(y) = \frac{\alpha y^{n-1}}{\theta^n}$$

$$F(y) = \int_0^y \frac{\alpha t^{n-1}}{\theta^n} dt = \left(\frac{y}{\theta} \right)^\alpha$$

$$F_{Y(n)}(y) = \left(\frac{y}{\theta} \right)^{n\alpha}, 0 \leq y \leq \theta$$

$$f_{Y(n)}(y) = \frac{n\alpha y^{n\alpha-1}}{\theta^{n\alpha}} 0 \leq y \leq \theta$$

So that $Y_{(n)}$ is also distributed as the power family with parameters $n\alpha$ and θ .

- a. $E(Y_{(n)}) = \frac{n\alpha\theta}{n\alpha+1} \neq \theta$.
- b. $\left(\frac{n\alpha+1}{n\alpha} \right) Y_{(n)}$ would be unbiased.
- c. $MSE(Y_{(n)}) = E[(Y_{(n)} - \theta)^2] = E(Y_{(n)}^2) - 2\theta E(Y_{(n)}) + \theta^2$
 $= \frac{n\alpha\theta^2}{n\alpha+2} - 2\theta \left(\frac{n\alpha\theta}{n\alpha+1} \right) + \theta^2$
 $= \frac{2\theta^2}{(n\alpha+1)(n\alpha+2)}$.

8.12 a. Let $S = \sqrt{S^2}$, where $\frac{(n-1)S^2}{\sigma^2}$ has a χ^2 distribution with $n-1$ degrees of freedom.

It is necessary to find $E(S)$. Let $X = \frac{(n-1)S^2}{\sigma^2}$. Then

$$f(x) = \frac{x^{(n-1)/2} e^{-x/2}}{\Gamma(\frac{n-1}{2}) 2^{(n-1)/2}} \quad \text{for } x > 0$$

The density function for $Y = S^2 = \frac{\sigma^2 X}{n-1}$ is obtained by the transformation method.

$$g(y) = f \left[\frac{(n-1)y}{\sigma^2} \right] \left| \frac{dy}{dx} \right| = \frac{\left[\frac{(n-1)}{\sigma^2} \right]^{(n-1)/2} y^{(n-1)/2-1} e^{-(n-1)y/2\sigma^2}}{\Gamma(\frac{n-1}{2}) 2^{(n-1)/2}}$$

Now

$$\begin{aligned} E(S) &= E(\sqrt{Y}) = \int_0^\infty y^{1/2} g(y) dy \\ &= \int_0^\infty y^{1/2} \frac{\left(\frac{n-1}{\sigma^2} \right)^{(n-1)/2} y^{(n-1)/2-1} e^{-(n-1)y/2\sigma^2}}{\Gamma(\frac{n-1}{2}) 2^{(n-1)/2}} dy \\ &= \frac{\left(\frac{n-1}{\sigma^2} \right)^{(n-1)/2} \Gamma(\frac{n}{2}) (2\sigma^2)^{n/2}}{\Gamma(\frac{n-1}{2}) 2^{(n-1)/2} (n-1)^{n/2}} \int_0^\infty y^{(n/2)-1} e^{-(n-1)y/2\sigma^2} dy \\ &= \frac{(n-1)^{-1/2} \Gamma(\frac{n}{2}) (\sigma^2)^{1/2}}{\Gamma(\frac{n-1}{2}) 2^{-1/2}} = \frac{\Gamma(\frac{n}{2}) 2^{1/2} \sigma}{\Gamma(\frac{n-1}{2}) \sqrt{n-1}} \end{aligned}$$

which is a biased estimator for σ .

b. In order to adjust S so that it is not biased, take

$$\hat{\sigma} = S \left(\frac{\Gamma(\frac{n-1}{2}) \sqrt{n-1}}{\Gamma(\frac{n}{2}) \sqrt{2}} \right)$$

8.14 $F_{Y(1)}(t) = P(Y_{(1)} \leq t) = 1 - P(Y_{(1)} > t) = 1 - [1 - F(t)]^n$

$$f_{Y(1)}(t) = n[1 - F(t)]^{n-1} f(t) = n \left[1 - \left(\frac{t}{\theta} \right) \right]^{n-1} \left(\frac{1}{\theta} \right)$$

$$E(Y_{(1)}) = \int_0^\theta nt \left[1 - \left(\frac{t}{\theta} \right) \right]^{n-1} \left(\frac{1}{\theta} \right) dt$$

Letting $w = \frac{t}{\theta}$, $dw = \frac{dt}{\theta}$, $t = \theta w$;

$$= \theta \int_0^1 nw(1-w)^{n-1} dw$$

$$= n\theta B(2, n) = \frac{n\theta \Gamma(2)\Gamma(n)}{\Gamma(n+2)}$$

$$= \frac{n\theta(n-1)!}{(n+1)!} = \frac{\theta}{(n+1)}$$

An unbiased estimator for θ is given by $(n+1)Y_{(1)}$.