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MATH 323, LOI

Solutions to Assignment #4

9.2

a. Since $E(Y_i) = \mu, i = 1, 2, \dots, n$, then

$$E(\hat{\mu}_1) = \frac{1}{2}(2\mu) = \mu \quad E(\hat{\mu}_2) = \frac{1}{4}\mu + \frac{(n-2)\mu}{2(n-2)} + \frac{1}{4}\mu = \mu \quad E(\hat{\mu}_3) = \frac{n\mu}{n} = \mu$$

b. Further,

$$V(\hat{\mu}_1) = \frac{1}{4}(2\sigma^2) = \frac{\sigma^2}{2} \quad V(\hat{\mu}_2) = \frac{2}{16}\sigma^2 + \frac{(n-2)\sigma^2}{4(n-2)^2} = \frac{\sigma^2}{8} + \frac{\sigma^2}{4(n-2)} \quad V(\hat{\mu}_3) = \frac{\sigma^2}{n}$$

Hence the efficiency of $\hat{\mu}_3$ relative to $\hat{\mu}_1$ is

$$\frac{V(\hat{\mu}_1)}{V(\hat{\mu}_3)} = \frac{\left(\frac{\sigma^2}{2}\right)}{\left(\frac{\sigma^2}{n}\right)} = \frac{n}{2}$$

and the efficiency of $\hat{\mu}_3$ relative to $\hat{\mu}_2$ is

$$\frac{V(\hat{\mu}_2)}{V(\hat{\mu}_3)} = \frac{\frac{\sigma^2}{8} + \frac{\sigma^2}{4(n-2)}}{\frac{\sigma^2}{n}} = \frac{n}{8} + \frac{n}{4(n-2)} = \frac{n^2}{8(n-2)}$$

9.4

We must first find the $V(\hat{\theta}_1)$ and $V(\hat{\theta}_2)$. Notice that the density of $Y_{(1)}$ is given by

$$nf(y)(1 - F(y))^{n-1} = \left(\frac{n}{\theta}\right) \left[1 - \left(\frac{y}{\theta}\right)\right]^{n-1} \text{ for } 0 \leq y \leq \theta.$$

Let $U_{(1)} = Y_{(1)}/\theta$. Then (using the transformation method) the density of $U_{(1)}$ is then given by $n(1-u)^{n-1}$ for $0 \leq u \leq 1$. That is, $U_{(1)}$ is distributed Beta($1, n$). Then

$$E(\theta_1) = E((n+1)Y_{(1)}) = E((n+1)U_{(1)}\theta) = \theta$$

and,

$$V(\theta_{(1)}) = V[(n+1)Y_{(1)}] = V((n+1)U_{(1)}\theta) = (n+1)^2\theta^2 V(U_{(1)}) = \frac{n\theta^2}{n+2}.$$

It was shown in Example 9.1 (or an equivalent calculation as above) yields

$$V\left[\frac{(n+1)Y_{(1)}}{n}\right] = \frac{\theta^2}{(n+2)}$$

Thus,

$$\text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)} = \frac{\left(\frac{\theta^2}{n+2}\right)}{\left(\frac{n\theta^2}{n+2}\right)} = \frac{1}{n^2}.$$

9.6 We have

$$V(\hat{\lambda}_1) = V\left[\frac{1}{2(Y_1+Y_2)}\right] = \left(\frac{1}{4}\right)\lambda = \frac{\lambda}{2}$$

$$V(\hat{\lambda}_2) = V(\bar{Y}) = \frac{\lambda}{n}.$$

Implying

$$\text{eff}(\hat{\lambda}_1, \hat{\lambda}_2) = \frac{2}{n}.$$

This suggest (as our intuition would) that using all of the data is much more efficient than using only two points.

- 9.10** Notice that $\hat{\sigma}_2^2$ doesn't change with n therefore we wouldn't expect it to be consistent. More precisely, $\lim_{n \rightarrow \infty} P(|\hat{\sigma}_2^2 - \sigma^2| > \epsilon) = P(|\hat{\sigma}_2^2 - \sigma^2| > \epsilon) > 0$.

- 9.12** Write

$$\frac{\sum (X_i - \bar{X})^2 + \sum (Y_i - \bar{Y})^2}{2n-2} = \frac{\sum (X_i - \bar{X})^2}{2n-2} + \frac{\sum (Y_i - \bar{Y})^2}{2n-2}$$

and consider only $\frac{\sum (X_i - \bar{X})^2}{2n-2} = \left(\frac{n}{2n-2}\right) \left(\frac{1}{n} \sum X_i^2 - \bar{X}^2\right)$. From Example 9.3, the quantity $\left(\frac{1}{n}\right) \sum X_i^2 - \bar{X}^2$ converges in probability to σ^2 , while $\frac{n}{2n-2} = \frac{1}{2 - \left(\frac{1}{n}\right)}$ converges to $\frac{1}{2}$ when $n \rightarrow \infty$. Hence $\sum \frac{(X_i - \bar{X})^2}{2n-2}$ converges in probability to $\frac{\sigma^2}{2}$. A similar argument shows that

$$\frac{\sum (Y_i - \bar{Y})^2}{2n-2} + \frac{\sigma^2}{2},$$

so that

$$\frac{\sum (X_i - \bar{X})^2 + \sum (Y_i - \bar{Y})^2}{2n-2} + \sigma^2.$$

- 9.14** Let $X_i = 1$ if trial i results in a success and $X_i = 0$ otherwise. Then for $i = 1, 2, \dots, n$, X_1, X_2, \dots, X_n constitutes a random sample from a distribution with mean p and variance $pq < \infty$. By the law of large numbers, $\bar{X} = \frac{Y}{n}$ is consistent for p .

- 9.18** a. We know the distribution of Y_i^2 is Chi squared 1. Then $\sum_{i=1}^n Y_i^2$ is Chi squared n . Refer to chapter 6 theorem 6.4.

- b. By example 9.2 (the law of large numbers), $W_n = \frac{1}{n} \sum_{i=1}^n Y_i^2$ converges to $E(Y_1^2) = 1$. To

prove this on our own, notice $E(W_n) = \frac{1}{n} \sum_{i=1}^n E(Y_i^2)/n = \frac{1}{n} \sum_{i=1}^n 1/n = 1$, and

$$V(W_n) = \frac{1}{n^2} V(\sum_{i=1}^n Y_i^2) = \frac{1}{n^2} \sum_{i=1}^n V(Y_i^2) = \frac{1}{n^2} \sum_{i=1}^n 2/n^2 = 2n/n^2 = 2/n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

- 9.20** a. Notice if $\epsilon > \theta$ then,

$$P(|Y_{(n)} - \theta| \leq \epsilon) = P(\theta - \epsilon \leq Y_{(n)} \leq \theta + \epsilon) = F(\theta + \epsilon) - F(\theta - \epsilon) = 1 - 0 = 1.$$

If $\epsilon < \theta$ then,

$$P(|Y_{(n)} - \theta| \leq \epsilon) = P(\theta - \epsilon \leq Y_{(n)} \leq \theta + \epsilon) = F(\theta + \epsilon) - F(\theta - \epsilon) = 1 - \left(\frac{\theta - \epsilon}{\theta}\right)^n.$$

- b. Evaluating,

$$\lim_{n \rightarrow \infty} 1 = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} 1 - \left(\frac{\theta - \epsilon}{\theta}\right)^n = 1, \quad \text{where } \epsilon > 0$$

Therefore, for any $\epsilon > 0$, $\lim_{n \rightarrow \infty} P(|Y_{(n)} - \theta| \leq \epsilon) = 1$. Hence $Y_{(n)}$ is a consistent estimator for θ .

(3)

9.24 \bar{Y} will converge in probability to $E(Y) = \mu_y$ provided (Y) is finite. Notice

$$E(Y) = \int_0^1 3y^3 dy = \left(\frac{3}{4}y^4\right]_0^1 = \frac{3}{4}$$

and,

$$E(Y^2) = \int_0^1 3y^4 dy = \frac{3}{5} \text{ (this step proves } V(Y) \text{ is finite).}$$

Thus, \bar{Y} converges in probability to $\frac{3}{4}$.

9.26 Notice that $E(Y^2) = \int_2^\infty 2 dy = \infty$, $V(Y)$ is not finite. Hence the law of large numbers cannot be applied to this problem.