

MATH 323 LO1

Solutions to Assignment #7

11.2 Calculate

$$\sum x_i = 720$$

$$\sum y_i^2 = 105,817$$

$$S_{xy} = \sum x_i y_i - \frac{1}{n} (\sum x_i) (\sum y_i) = 54,243$$

$$S_{xx} = \sum x_i^2 - \frac{1}{n} (\sum x_i)^2 = 54,714$$

$$\sum y_i = 721$$

$$\sum x_i y_i = 106,155$$

$$\sum x_i^2 = 106,554$$

$$n = 10$$

Then

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{54,243}{54,714} = .9913916$$

and

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = 72.1 - .99(72) = .7198048.$$

The least squares straight line is $\hat{y} = .72 + .99x$, and the expected change in y for a one-unit change in x is estimated as $\hat{\beta}_1 = .99$. When $x = 100$, the best estimate of y is $\hat{y} = .72 + .99(100) = 99.72$.

11.6 We need to minimize SSE = $\sum_{i=1}^n [y_i - \hat{\beta}_1 x_i]^2$. Consider

$$\begin{aligned} \frac{d \text{SSE}}{d \hat{\beta}_1} &= - \sum_{i=1}^n 2 [y_i - \hat{\beta}_1 x_i] x_i = 0 \\ &= -2 \left[\sum_{i=1}^n (x_i y_i - \hat{\beta}_1 x_i^2) \right] = 0 \\ &\Rightarrow \sum_{i=1}^n x_i y_i - \hat{\beta}_1 \sum_{i=1}^n x_i^2 = 0 \end{aligned}$$

Implying

$$\hat{\beta}_1 = \frac{\sum x_i y_i}{\sum x_i^2}$$

11.8 a. We calculate:

$$\sum_{i=1}^5 x_i = 102$$

$$\sum_{i=1}^5 x_i y_i = 894.4$$

$$S_{xy} = -425.48$$

Implying

$$\sum_{i=1}^5 x_i^2 = 3940$$

$$\sum_{i=1}^5 y_i^2 = 949.99$$

$$S_{xx} = 1859.2$$

$$\sum_{i=1}^5 y_i = 64.7$$

$$n = 5$$

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = -.229$$

and

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = \frac{64.7}{5} - (-.229) \left(\frac{102}{5} \right) = 17.611$$

b. The data and least squares line are plotted in Figure 11.4.

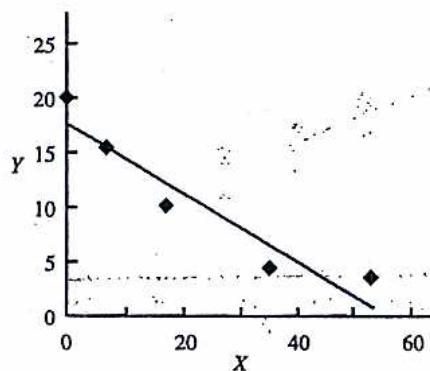


Figure 11.4

c. The estimate of $E(Y)$ when $x = 20$ is $\hat{y} = 17.611 - .229(20) = 13.031$.

11.12 a. Calculate

$$\sum x_i = 720$$

$$\sum x_i^2 = 49,200$$

$$S_{xy} = -1900.0$$

Then

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = -.31667$$

and

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = 46.00$$

b. The least squares line shown in Figure 11.5 is $\hat{y} = 46.0 - .317x$.

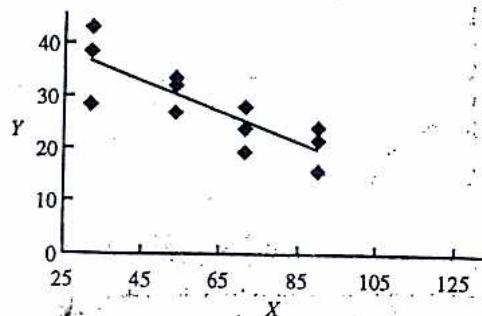


Figure 11.5

c. Refer to Exercise 11.11.

$$\begin{aligned} SSE &= S_{yy} - \hat{\beta}_1 S_{xy} \\ &= 9540 - \frac{(324)^2}{12} + (.31667) \left[17,540 - \frac{(720)(324)}{12} \right] = 190.333 \end{aligned}$$

and

$$s^2 = \frac{SSE}{n-2} = \frac{190.333}{10} = 19.0333$$

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11.16 The likelihood function is

$$L = \frac{1}{(2\pi)^{n/2} \sigma^n} \prod_{i=1}^n \exp \left\{ -\frac{1}{2\sigma^2} [y_i - E(y_i)]^2 \right\}$$

$$= K \exp \left[-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2 \right]$$

so that

$$\ln L = \ln K - \frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2.$$

In order to maximize $\ln L$ with respect to β_0 and β_1 , it is necessary to maximize

$$-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2$$

which implies choosing $\hat{\beta}_0$ and $\hat{\beta}_1$ to minimize $\sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 = \text{SSE}$. Hence the maximum likelihood estimators of β_0 and β_1 are identical to the least squares estimators.

11.22a.

$\sum_{i=1}^6 x_i = 323.4$	$\sum_{i=1}^6 y_i = 42.6$	$\sum_{i=1}^6 x_i y_i = 2495.08$
$\sum_{i=1}^6 x_i^2 = 19,111.95$	$\sum_{i=1}^6 y_i^2 = 326.06$	$S_{xy} = 198.94$
$S_{xx} = 1680.69$	$S_{yy} = 23.6$	
$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{198.94}{1680.69} = .118$		
$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = \frac{42.6}{6} - .118 \left(\frac{323.4}{6} \right) = .72$		

b. $\text{SSE} = S_{yy} - \hat{\beta}_1 S_{xy} = 23.6 - (.118)(198.94) = .125$
and
 $s^2 = \frac{\text{SSE}}{n-2} = \frac{.052}{6-2} = .013$

A 95% confidence interval for β_1 is

$$\hat{\beta}_1 \pm t_{.025, 4} s \sqrt{c_{11}}$$

$$.118 \pm 2.776 \sqrt{.013} \sqrt{.00059}$$

$$.118 \pm .008$$

(4)

11.24 Restricting ourselves to Ω_0 , we find the likelihood function to be

$$L(\Omega_0) = \frac{1}{(2\pi)^{n/2}\sigma^n} \exp \left[-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \beta_0)^2 \right]$$

One may verify that maximum likelihood estimates of β_0 and σ^2 are

$$\hat{\beta}_0 = \bar{Y} \quad \text{and} \quad \hat{\sigma}_0^2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n} \quad \text{so that} \\ L(\hat{\Omega}_0) = \frac{1}{(2\pi)^{n/2} (\hat{\sigma}_0^2)^{n/2}} e^{-n/2}.$$

For β_1 in the unrestricted space Ω , the likelihood function is given in the solution to Exercise 11.16, and the maximum likelihood estimates of β_0 and β_1 are the least squares estimates

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x} \quad \text{and} \quad \hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$$

The maximum likelihood estimate of σ^2 is $\hat{\sigma}^2 = \frac{\sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2}{n}$ so that

$$L(\hat{\Omega}) = \frac{1}{(2\pi)^{n/2} (\hat{\sigma}^2)^{n/2}} e^{-n/2}.$$

Hence

$$\lambda^{2/n} = \frac{\hat{\sigma}^2}{\hat{\sigma}_0^2} = \frac{\sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2}{S_{yy}} = \frac{SSE}{S_{yy}}$$

Then

$$\begin{aligned} S_{yy} &= \sum_{i=1}^n [y_i - \bar{y} - \hat{\beta}_1(x_i - \bar{x}) + \hat{\beta}_1(x_i - \bar{x})]^2 \\ &= \sum_{i=1}^n [y_i - \bar{y} - \hat{\beta}_1(x_i - \bar{x})]^2 + \hat{\beta}_1^2 S_{xx} + 2\hat{\beta}_1 S_{xy} - 2\hat{\beta}_1^2 S_{xx} \\ &= SSE + 2\hat{\beta}_1 S_{xy} - \hat{\beta}_1^2 S_{xx} \end{aligned}$$

But $\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$ so that

$$S_{yy} = SSE + 2 \frac{S_{xy}^2}{S_{xx}} - \frac{S_{xy}^2}{S_{xx}} = SSE + \frac{S_{xy}^2}{S_{xx}}.$$

Now

$$\lambda^{2/n} = \frac{SSE}{SSE + \left(\frac{S_{xy}^2}{S_{xx}}\right)} = \frac{1}{1 + \left(\frac{T^2}{n-2}\right)}$$

where $T^2 = \frac{S_{xy}^2}{S_{xx} \left(\frac{SSE}{n-2}\right)} = \frac{\beta_1^2 \sum_i (x_i - \bar{x})^2}{\frac{SSE}{n-2}}$. Hence as λ gets small, T^2 will get large (either positively or negatively), and rejecting H_0 for large or small values of T will be equivalent to rejecting H_0 for small values of λ . Note that

$$T = \frac{\hat{\beta}_1}{\sqrt{\frac{\hat{\sigma}^2}{\sum_{i=1}^n (x_i - \bar{x})^2}}}$$

is the t test given in Section 11.6.