UNIVERSITY OF CALGARY

DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 323 — Quiz no.5 — Sections T01,T02 — April 10, 2007 SOLUTIONS

TIME: 30 minutes

Marks

NAME: ~~Marking Key~~

5 1. Let X have a Binominal distribution with $n = 5$ and p unknown.

3 (a) Use the Neyman-Pearson lemma to find the most powerful test of the null hypothesis $H_0 : p = 1/2$ against the alternative hypothesis $H_1 : p = 2/3$ at level $\alpha = 6/32$.

$$L(x,p) = \binom{5}{x} p^x (1-p)^{5-x} \leftarrow \tfrac{1}{2}$$

By the N-P lemma, the MP test rejects H_0 when $\underbrace{\frac{L(x,1/2)}{L(x,2/3)} \text{ is too small}}_{1/2}$

OR when $\dfrac{\binom{5}{x}(1/2)^x(1/2)^{5-x}}{\binom{5}{x}(2/3)^x(1/3)^{5-x}} = \dfrac{(1/2)^5}{(2/3)^5\ (2/3)(1/3)^x} = \text{const.} \times \left(\frac{1}{2}\right)^x$ is too small.

OR when X is too large. $\leftarrow 1$

So the MP test rejects when $X \geq C$, where C is determined by $P_{p=1/2}[X \geq C] = \frac{6}{32}$

By trial-and-error, $P_{p=1/2}[X \geq 4] = (\frac{1}{2})^5 + 5(\frac{1}{2})^4(\frac{1}{2}) = \frac{6}{32}$. $\leftarrow \frac{1}{2}$ So MP level $\alpha = 6/32$ $\underbrace{\text{test is: reject } H_0 \text{ when } X \geq 4}_{1/2}$

2 (b) Find the power of the test of part (a) against the alternative $p = 2/3$.

The power at $p = 2/3$ is

$$P_{p=2/3}[\text{Reject } H_0] = P_{p=2/3}[X \geq 4] \leftarrow 1$$

$$= P_{p=2/3}[X=5] + P_{p=2/3}[X=4] = \left(\frac{2}{3}\right)^5 + 5\left(\frac{2}{3}\right)^4 \frac{1}{3} \leftarrow \frac{1}{2}$$

$$= \frac{32}{243} + \frac{80}{243} = \underline{\underline{\frac{112}{243}}} \leftarrow \frac{1}{2}$$

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2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with unknown mean $\lambda > 0$. Find the form of the level-α Likelihood Ratio Test of the null hypothesis $H_0 : \lambda = 1$ against the composite alternative hypothesis $H_1 : \lambda \neq 1$. (Simplify the form of the test statistic as much as possible.)

$$L(\lambda) = \prod_{i=1}^{n}\left[e^{-\lambda}\frac{\lambda^{x_i}}{x_i!}\right] = e^{-n\lambda}\frac{\lambda^{\Sigma x_i}}{\prod x_i!} \leftarrow 1$$

The L-R test rejects H_0 when $\dfrac{L(1)}{L(\hat\lambda)}$ is too small, $\leftarrow 1$

where $\hat\lambda$ is the MLE of λ.

To find this, note $\log L(\lambda) = -n\lambda + (\Sigma x_i)\log\lambda - \Sigma\log(x_i!)$,

and solve $\frac{d}{d\lambda}\log L(\lambda) = -n + \frac{\Sigma x_i}{\lambda} = 0$ to obtain

the MLE $\hat\lambda = \frac{\Sigma x_i}{n} = \bar{x}$. $\leftarrow 1$

So the L-R test rejects when $\dfrac{e^{-n}\,1^{(\Sigma x_i)}}{\prod x_i!}\;\dfrac{\prod x_i!}{e^{-n\bar{x}}\;\bar{x}^{(n\bar{x})}}$ too small $\leftarrow 1$

$$\left.\begin{array}{l}\text{OR when } \dfrac{e^{n\bar{x}}}{\bar{x}^{n\bar{x}}} \text{ too small.}\\[2ex] \text{OR when } n\bar{x}(1-\log\bar{x}) \text{ is too small.}\end{array}\right\} \leftarrow 1$$

(Give full credit for getting this far, even though very good students will be able to deduce that L-R test rejects when $\bar{X} \leq C_1$ or $\geq C_2$, and will be able to write down two sets of side conditions to determine C_1 and C_2 for achievable α's.)

10 marks total