

UNIVERSITY OF CALGARY

DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 323 — Quiz no.5 — Section T03 — April 10, 2007

TIME: 30 minutes

SOLUTIONS

NAME: ~~Marking Key~~

Markes

- 5 1. Let X have a Binominal distribution with $n = 4$ and p unknown.
 3 (a) Use the Neyman-Pearson lemma to find the most powerful test of the null hypothesis $H_0 : p = 1/3$ against the alternative hypothesis $H_1 : p = 1/4$ at level $\alpha = 16/81$.

$$L(x, p) = \binom{4}{x} p^x (1-p)^{4-x} \leftarrow (1/2)$$

By the N-P lemma, the MP test rejects H_0 when $\frac{L(x, 1/3)}{L(x, 1/4)}$ is too small

OR when $\frac{\binom{4}{x} (\frac{1}{3})^x (\frac{2}{3})^{4-x}}{\binom{4}{x} (\frac{1}{4})^x (\frac{3}{4})^{4-x}} = \left(\frac{2/3}{1/4}\right)^x \frac{[(1/3)/(2/3)]^x}{(1/4)/(3/4)^x}$ is too small

OR when $(\frac{3}{2})^x$ is too small, OR when X is too small $\leftarrow (1)$

So the MP test rejects H_0 when $X \leq C$, where C is determined by $P_{p=1/3} [X \leq C] = \frac{16}{81}$

By inspection, $P_{p=1/3} [X=0] = (\frac{2}{3})^4 = \frac{16}{81} \leftarrow (1/2)$, so the MP level $\alpha = 16/81$ test is: Reject H_0 when $X=0$

- 2 (b) Find the power of the test of part (a) against the alternative $p = 1/4$.

The power at $p=1/4$ is

$$P_{p=1/4} [\text{Reject } H_0] = P_{p=1/4} [X=0] = \left(\frac{3}{4}\right)^4 = \underline{\underline{\frac{81}{256}}}$$

5 2. Let X_1, X_2, \dots, X_n be a random sample from an exponential distribution with density function

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & \text{if } 0 < x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

with unknown parameter $\beta > 0$. Find the form of the level- α Likelihood Ratio Test of the null hypothesis $H_0 : \beta = 1$ against the composite alternative hypothesis $H_1 : \beta \neq 1$. (Simplify the form of the test statistic as much as possible.)

$$L(\beta) = \prod_{i=1}^n \left[\frac{1}{\beta} e^{-x_i/\beta} \right] = \frac{1}{\beta^n} e^{-\sum x_i/\beta} \quad \leftarrow \textcircled{1}$$

$x_i > 0, i=1, \dots, n$
 $\beta > 0$

The L-R test rejects H_0 when $\frac{L(1)}{L(\hat{\beta})}$ is too small, $\leftarrow \textcircled{1}$
where $\hat{\beta}$ is the MLE of β .

Given $x_1 > 0, \dots, x_n > 0$, solve $\frac{\partial}{\partial \beta} \log L(\beta) = 0$

OR $\frac{\partial}{\partial \beta} \left[-n \log \beta - \sum x_i/\beta \right] = 0$ OR $-\frac{n}{\beta} + \frac{\sum x_i}{\beta^2} > 0$

to obtain $\hat{\beta} = \frac{\sum x_i}{n} = \bar{x}$. $\leftarrow \textcircled{2}$

So the L-R test rejects H_0 when $\frac{e^{-\sum x_i}}{\frac{1}{\bar{x}^n} e^{-n\bar{x}/\bar{x}}} = \bar{x}^n e^{-n\bar{x}}$ is too small $\leftarrow \textcircled{3}$

OR when $\bar{x}^n e^{-n\bar{x}}$ is too small
or when $n \log \bar{x} - n\bar{x}$ is too small $\leftarrow \textcircled{4}$

[Give full credit for the above, although some students might deduce that H_0 is rejected when $\bar{x} \leq c_1$ or $\bar{x} > c_2$ and ~~that~~ even write down the side conditions determining c_1, c_2 from α].

10 marks total