Spring '05

MATH 331 # 1 (Solution)

18(a)/37 $\quad (Ax^2y + 2y^2)\,dx + (x^3 + 4xy)\,dy = 0$

is exact if

$$\frac{\partial M}{\partial y} = Ax^2 + 4y = \frac{\partial N}{\partial x} = 3x^2 + 4y$$

i.e if $\quad A = 3.$

Then solve $\int 3x^2y + 2y^2\,\partial x + \int x^3 + 4xy\,\partial y = c$

$$x^3y + 2y^2x + \cancel{x^3y} + 2xy^2 = c$$

or $\underline{x^3y + 2y^2x + 2xy^2 = C}$

24/37 $\quad [y + x(x^2+y^2)^2]\,dx + [y(x^2+y^2)^2 - x]\,dy = 0$

Mult. $\frac{1}{x^2+y^2}$, we get

$$\left[\frac{y}{x^2+y^2} + x(x^2+y^2)\right]dx + \left[y(x^2+y^2) - \frac{x}{x^2+y^2}\right]dy = 0$$

$$\therefore \frac{\partial M}{\partial y} = \frac{(x^2+y^2).1 - y.2y}{(x^2+y^2)^2} + 2xy = \frac{x^2-y^2}{(x^2+y^2)^2} + 2xy = \frac{\partial N}{\partial x}. \therefore \text{Exact.}$$

Hence, by grouping + integrating,

$$\left[\frac{y}{x^2+y^2} + xy^2\right]dx + x^3dx + y^3dy + \left[yx^2 - \frac{x}{x^2+y^2}\right]dy = 0$$

the solution is.

$$\left[\frac{\cancel{y}}{\cancel{y}}\tan^{-1}\frac{x}{y} + \frac{x^2y^2}{2}\right] + \frac{x^4}{4} + \frac{y^4}{4} \cancel{- \frac{x}{x}\tan^{-1}\frac{x}{y}} = C$$

22/47. (a). $(x+2y)\,dx + (2x-y)\,dy = 0$ is exact, since $\frac{\partial M}{\partial y} = 2 = \frac{\partial N}{\partial x}$

Integrating, we get the solution:

$$\frac{x^2}{2} + 2yx - \frac{y^2}{2} = c$$

The eq. is homogeneous. Set $y = vx$. Then

$$y' = v'x + v = \frac{x+2y}{y-2x} = \frac{x(1+2v)}{x(v-2)} = \frac{1+2v}{v-2}$$

$$\therefore\ v'x = \frac{1+2v}{v-2} - v = \frac{1+4v-v^2}{v-2}$$

$$\therefore\ \frac{v-2}{1+4v-v^2}\,dv = \frac{dx}{x}.$$

$$\therefore\ -\frac{1}{2}\ln|1+4v-v^2| = \ln x + \ln k.$$

$$\therefore\ \ln|1+4v-v^2|x^2 = \ln c.$$

$$\therefore\ x^2 + 4xy - y^2 = c.$$

(b). $(3x-y)\,dx - (x+y)\,dy = 0$ is exact, since $\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x}$.

Integrating

$$\frac{3x^2}{2} - yx - \frac{y^2}{2} = c \quad \text{or} \quad \underline{3x^2 - 2yx - y^2 = c}$$

Similar to (a), if you use $y = vx$.

19/56 $\frac{dy}{dx} - \frac{2}{x}y = 2x^3, \quad y(2)=8.$

$$I.F. = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

The general solution is

$$x^{-2}y = \int 2x^3 . x^{-2}dx = \int 2x\,dx = x^2 + c$$

$$y = x^2(x^2+c).$$

Using $y(2)=8$, we get $8 = 4(4+c) \therefore c = 2-4 = -2$

$$\therefore y = x^2(x^2-2).$$

24/56 Linear eq. $\frac{dx}{dt} - x = \sin 2t, \quad x(0)=0$

$$I.F. = e^{-\int 1\,dt} = e^{-t}$$

The general solution is

$$e^{-t}x = \int e^{-t}\sin 2t\,dt \quad \text{(by parts)}$$

$$I = \int e^{-t}\sin 2t\,dt = -e^{-t}\sin 2t + \int e^{-t}2\cos 2t\,dt$$

$$= -e^{-t}\sin 2t + 2\left[-e^{-t}\cos 2t - \int -e^{-t}.-2\sin 2t\,dt\right]$$

$$= -e^{-t}\sin 2t - 2e^{-t}\cos 2t - 4I.$$

$$5I = -e^{-t}(\sin 2t + 2\cos 2t).$$

Hence the general solution is

$$e^{-t}x = -\frac{1}{5}e^{-t}(\sin 2t + 2\cos 2t) + c$$

$$x = -\frac{1}{5}\cdot(\sin 2t + 2\cos 2t) + ce^{t}$$

Using $x(0)=0$, we get $0 = -\frac{1}{5}(2) + ce^{0} \therefore c = \frac{2}{5}$

$$\therefore x = -\frac{1}{5}(\sin 2t + 2\cos 2t) + \frac{2}{5}e^{t}$$

19/60. Solve $4xy \frac{dy}{dx} = y^2+1$, $y(2)=1$

$$\frac{4y}{y^2+1}\,dy = \frac{dx}{x} \qquad \text{Separable}$$

$$2\ln(y^2+1) = \ln|x| + c$$

$$\text{or } \ln\frac{(y^2+1)^2}{x} = c \quad \text{or} \quad \frac{(y^2+1)^2}{x} = k \qquad (k = e^c)$$

Since $y(2)=1$, we have $k = \frac{2^2}{2} = 2$.

$$\therefore (y^2+1)^2 = 2x.$$

13/60. Solve $\frac{dy}{dx} + \frac{6x^2}{x^3+1}\,y = \frac{6x^2}{x^3+1}$ Linear eq.

$$I.F. = e^{\int \frac{6x^2}{x^3+1}dx} = e^{\ln|x^3+1|} = x^3+1.$$

The general solution is

$$(x^3+1)y = \int \frac{6x^2}{\cancel{x^3+1}}\cdot\cancel{(x^3+1)}\,dx = \frac{6x^3}{3}+c = 2x^3+c$$

$$\therefore y = \frac{2x^3+c}{(x^3+1)}.$$

10/78 Find Orthogonal Trajectories of

$$x^2 - y^2 = cx^3.$$

First: $2x - 2y \cdot \frac{dy}{dx} = 3cx^2 = 3\,\frac{x^2-y^2}{x^3}\,x^2 = 3\,\frac{x^2-y^2}{x}$ eliminate "c".

$$-2y\,\frac{dy}{dx} = -2x + 3x - \frac{3y^2}{x} = \cancel{x} - \frac{3y^2}{x} = \frac{x^2-3y^2}{x}$$

$$\therefore \frac{dy}{dx} = \frac{x^2-3y^2}{-2xy}$$

$\therefore$ The O.T. are given by

$$\frac{dy}{dx} = \frac{-1}{\frac{x^2-3y^2}{-2xy}} = \frac{2xy}{x^2-3y^2}$$ homogeneous eq.

Let $y = ux$, (or $u = y/x$)

Then $$x\frac{du}{dx} + u = \frac{x^2 \cdot 2u}{x^2(1-3u^2)} = \frac{2u}{1-3u^2}$$ (no "x" term)

$$x\frac{du}{dx} = \frac{2u}{1-3u^2} - u = \frac{2u - u(1-3u^2)}{1-3u^2}$$

$$= \frac{3u^3+u}{1-3u^2}$$

$$\therefore \int \frac{1-3u^2}{3u^3+u}\,du = \int \frac{dx}{x}$$

$$\therefore \int \frac{1-3u^2}{u(3u^2+1)}\,du = \ln|x| + c$$

$$\frac{1-3u^2}{u(3u^2+1)} = \frac{A}{u} + \frac{Bu+C}{3u^2+1}$$

$A = 1$ & $B = -6$, $C = 0$

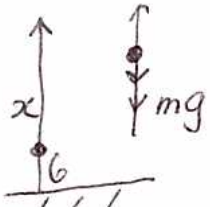
$$\therefore \int \frac{1}{u} - \frac{6u}{3u^2+1}\,du = \ln|x| + C$$

$$\ln|u| - \ln(3u^2+1) = \ln|x| + C.$$

$$\ln\frac{u}{(3u^2+1)x} = C$$

$$\frac{u}{(3u^2+1)x} = e^C = K \quad \therefore \frac{y/x}{(3y^2+1)x} = K, \text{ or } \frac{y}{3y^2+x^2} = K.$$

3/88.



Measure x upwards from ground level.
The forces on the upward moving particle are $-mg - \frac{v}{64} = -\frac{3}{4} - \frac{v}{64}$.

Since $mg = \frac{3}{4}$

$$\therefore m = \frac{3}{4g} = \frac{3}{4 \cdot 32} = \frac{3}{128}$$

By Newton's law:

$$m v' = -\frac{3}{4} - \frac{v}{64}$$

$$\frac{3}{128}\frac{dv}{dt} = \frac{-3}{4} - \frac{v}{64}$$

$$\frac{dv}{dt} = \frac{128}{3}\left(-\frac{3}{4} - \frac{v}{64}\right) = -32 - \frac{2v}{3}$$

But $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$

$$v\frac{dv}{dx} = -\left(32 + \frac{2v}{3}\right), \quad v(6) = 20 \text{ (initial condition)}$$

$$v\frac{dv}{dx} = -\frac{96+2v}{3} \qquad \text{Separable}$$

$$\int \frac{v\,dv}{96+2v} = \int -\frac{dx}{3}$$

$$2v+96 \overline{\big) \, v \,} \quad \frac{1}{2}, \qquad \frac{v+48}{-48}$$

You can also use $\int_{20}^{0} \frac{v\,dv}{96+2v} = -\int_{6}^{x} \frac{dx}{3}$

$$\int \left(\frac{1}{2} - \frac{48}{96+2v}\right) dv = \int -\frac{dx}{3}$$

$$\frac{1}{2}v - \frac{48}{2}\ln(96+2v) = -\frac{x}{3} + c \quad \text{——— (1)}$$

When $x = 6$, $v = 20$,

$$\therefore \quad 10 - 24\ln 136 = -2 + c$$

$$\therefore c = 12 - 24\ln 136$$

Subst for c in (1),

$$\frac{v}{2} - 24\ln(96+2v) = -\frac{x}{3} + 12 - 24\ln 136$$

The highest point is reached when $v = 0$:

$$-24\ln 96 = -\frac{x}{3} + 12 - 24\ln 136$$