Spring '05 MATH 331 (L20) ①

Assignment 4 (Solution)

#4/p118. $f(tx, ty) = t^m f(x,y)$

Let $u = tx$, $v = ty$. Then $f(tx,ty) = f(u,v) = t^m f(x,y)$.

From Chain rule, (and $\frac{\partial u}{\partial t} = x$, $\frac{\partial v}{\partial t} = y$), we get:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} = \frac{\partial f}{\partial u}x + \frac{\partial f}{\partial v}y = mt^{m-1}f(x,y)$$

i.e. $$\frac{\partial}{\partial t}f = \left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right)f = mt^{m-1}f(x,y)$$

From Chain rule again,

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}f\right) = \left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right)\left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right)f$$

$$= \left(x^2\frac{\partial^2}{\partial u^2} + 2xy\frac{\partial^2}{\partial u\partial v} + y^2\frac{\partial^2}{\partial v^2}\right)f$$

$$= m(m-1)t^{m-2}f(x,y).$$

Set $t=1$.

Then $u = x$ & $v = y$, and so

$$\left(x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x\partial y} + y^2\frac{\partial^2}{\partial y^2}\right)f = m(m-1)f(x,y).$$

#9/p119 $z = \sin(x+ct) + \cos(2x+2ct)$.

Then $\frac{\partial z}{\partial t} = \cos(x+ct)\cdot c - \sin(2x+2ct)\cdot 2c$ (Chain rule)

$\therefore \frac{\partial^2 z}{\partial t^2} = -\sin(x+ct)\cdot c^2 - \cos(2x+2ct)\cdot 4c^2$ —— ①

Also, $\frac{\partial z}{\partial x} = \cos(x+ct) - \sin(2x+2ct)\cdot 2$

$\frac{\partial^2 z}{\partial x^2} = -\sin(x+ct) - \cos(2x+2ct)\cdot 4$ —— ②

Hence $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$. (from ① $= c^2$②).

#13 (a) If $g(r,\theta) = r^n \cos n\theta$ (n is +ve integer). ②

then $g_r = n r^{n-1} \cos n\theta$

$g_{rr} = n(n-1) r^{n-2} \cos n\theta$

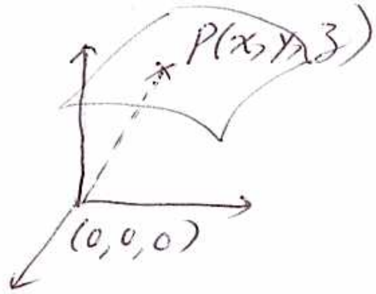
$g_\theta = -n r^n \sin n\theta$ & $g_{\theta\theta} = -n^2 r^n \cos n\theta$

Thus, $g_{rr} + \frac{1}{r} g_r + \frac{1}{r^2} g_{\theta\theta}$

$= r^{n-2} \cos n\theta \left(n(n-1) + n - n^2 \right) = 0.$

(b) Similarly, if $g(r,\theta) = r^n \sin n\theta$.

#9/p140. Let any pt $P(x,y,z)$ be on the surface $z^2 - xy = 1$.



Then the distance (s) from O to P is:

$u = s^2 = x^2 + y^2 + z^2$

Let $g(x,y,z) \equiv z^2 - xy - 1 = 0.$ —— (A)

From Lagrange Multiplier method for

$u = \lambda g,$

we get $u_x = \lambda g_x$, $u_y = \lambda g_y$ and $u_z = \lambda g_x$ and

$\therefore$ $2x = -\lambda y$ —— ①

$2y = -\lambda x$ —— ②

$2z = 2\lambda z$ —— ③

From ① & ②: $\dfrac{2x}{2y} = \dfrac{-\lambda y}{-\lambda x}$ $\therefore$ $x^2 = y^2$ ie $x = \pm y$.

From ③: $2z(1-\lambda) = 0$ $\therefore$ $z = 0$ or $\lambda = 1$.

Case (i): If $z = 0$, $x = y$, then from (A), we get $-y^2 - 1 = 0$ or $y^2 = -1$ (not possible)

" (ii): If $z = 0$, $x = -y$, then $y^2 = 1$ ie $y = \pm 1$.

$\therefore$ $u(-1, 1, 0) = 2$ and $u(1, -1, 0) = 2$

Case (iii): If $\lambda = 1$ (i.e. $z \neq 0$),

then ① $\rightarrow$ $2x = -y$

② $\rightarrow$ $2y = -x$ $\quad \therefore 2x = -\left(-\frac{x}{2}\right)$ or $4x = x$

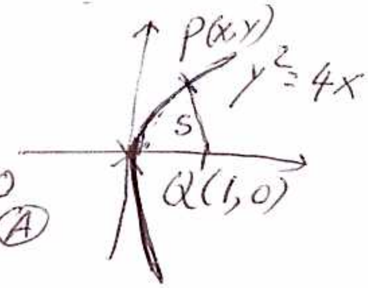
$\therefore x = 0$

& $y = 0$ $\quad$ From Ⓐ : $z = \pm 1$

$\therefore u(0,0,\pm 1) = 0 + 0 + 1 = 1$ (Compare with Case (ii)).

Hence there are 2 pts $(0,0,\pm 1)$ which are closest to the origin and their distance is 1.

#11 Distance PQ (is s):



$u = s^2 = (x-1)^2 + y^2$ & $g \equiv y^2 - 4x = 0$ Ⓐ

Using Lagrange Multiplier,

$u_x = \lambda g_x$ & $u_y = \lambda g_y$,

$2(x-1) = -4\lambda$ & $2y = 2\lambda y$ $\quad \therefore 2y(1-\lambda) = 0$

$\therefore y = 0$ or $\lambda = 1$

Case (i) $y = 0$. Then $x = 0$ (from Ⓐ).

$u(0,0) = 1$

" (ii) $\lambda = 1$. Then $2(x-1) = -4$ $\quad \therefore x = -1$ & $y^2 = 4x = -4$ (impossible)

Hence shortest dist. is given in Case (i).

#12/p140 $\quad f = x+y+z$ subject to $g \equiv x^2+y^2+z^2-1=0$. (A)

(i) Inside: $g \equiv x^2+y^2+z^2-1<0$

$$f_x = 1,\ f_y = 1,\ f_z = 1 \quad \therefore \text{ no c.p.}$$

(ii) On the boundary $g=0$, we consider L. Multiplier:

$$f = \lambda g$$

$$f_x = \lambda g_x,\quad f_y = \lambda g_y,\quad f_z = \lambda g_z$$

$$\therefore 1 = 2\lambda x,\ 1 = 2\lambda y,\ 1 = 2\lambda z$$

$$\therefore x = y = z \left(= \frac{1}{2\lambda}\right).$$

$\therefore$ From (A), $\quad 3x^2 = 1 \quad \therefore x = \pm\frac{1}{\sqrt{3}}$

$y = \pm\frac{1}{\sqrt{3}}$

$z = \pm\frac{1}{\sqrt{3}}$.

8 C.P. are $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

$\therefore$ Global max $f = \frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

" $\quad$ min $f = -\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}} = -\sqrt{3}$.

<u>Prob. 3.</u> $\quad f = x^2+2y^2-x$

(i) Inside $D \equiv x^2+y^2<1$ $\qquad \left.\begin{aligned} f_x &= 2x-1 = 0 \\ f_y &= 4y = 0 \end{aligned}\right\}$ for C.P.

$\therefore x = \frac{1}{2},\ y = 0$ is inside D.

$\therefore f = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$. (Min. Value)

(ii) On the boundary $D \equiv x^2+y^2-1=0$ i.e. $y^2 = 1-x^2$

You can also use L. Multiplier Method

$\therefore f = x^2+2(1-x^2)-x = 2-x^2-x$

$f' = -2x-1 = 0$, for C.P $\quad \therefore x = -\frac{1}{2}$ $\left(y^2 = 1-\frac{1}{4} = \frac{3}{4}\right)$

$\therefore f = 2-\frac{1}{4}+\frac{1}{2} = 2\frac{1}{4}$ (Max.)

Prob. 4 $f = x + z$ subj. to $g \equiv x^2 + y^2 + z^2 - 1 = 0$

$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad f_z = \lambda g_z$ (or $\nabla f = \lambda \nabla g$.)

$1 = 2\lambda x$ ①

$0 = 2\lambda y$ ② $\longrightarrow \lambda = 0$ or $y = 0$

$1 = 2\lambda z$ ③

$x = z \left(= \frac{1}{2\lambda}\right)$

Case (i) $\lambda = 0$ then $1 = 0$ not impossible (from ①)

(ii) $y = 0$ then $x = z$

$\therefore$ From $g = 0$, we get $2x^2 = 1$

$\therefore x = \pm \frac{1}{\sqrt{2}}$

$z = \pm \frac{1}{\sqrt{2}}$

Hence 2 C.P. are $\left(+\frac{1}{\sqrt{2}}, 0, +\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$.

$f\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$f\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$.

Prob 5 $f = x^2 + y^2 + 2z^2$ subj. to $g \equiv 2x^2 + y^2 + z^2 - 1 = 0$ Ⓐ

From $\nabla f = \lambda \nabla g$, we get

$2x = 4\lambda x \quad \therefore 2x(1 - 2\lambda) = 0 \quad \therefore x = 0$ or $\lambda = \frac{1}{2}$

$2y = 2\lambda y \quad\quad 2y(1 - \lambda) = 0 \quad\quad y = 0$ " $\lambda = 1$

$4z = 2\lambda z \quad\quad 2z(2 - \lambda) = 0 \quad\quad z = 0$ " $\lambda = 2$

Case (i) $x = 0, y = 0, z = 0$ impossible (Ⓐ is not satisfied)

(ii) $x = 0, y = 0, \lambda = 2 \Rightarrow$ Using Ⓐ, $\therefore z = \pm 1$.

$\therefore$ 2 C.P. $(0, 0, \pm 1)$

(iii) $x = 0, \lambda = 1, z = 0 \Rightarrow$ $y = \pm 1$

$\therefore$ 2 C.P. $(0, \pm 1, 0)$

(iv) $\lambda = \frac{1}{2}, y = 0, z = 0 \Rightarrow$ $x = \pm \frac{1}{\sqrt{2}}$

$\therefore$ 2 C.P. $\left(\pm \frac{1}{\sqrt{2}}, 0, 0\right)$.

$f(0,0,\pm 1) = 2$

$f(0,\pm 1,0) = 1$

$f(\pm\frac{1}{\sqrt{2}},0,0) = \frac{1}{2}$

Hence max f is 2 & min. f is $\frac{1}{2}$.

Prob 6 $\quad f = xyz$ subject to $g = x^2+y^2+z^2-1 = 0$ (A)

$\nabla f = \lambda \nabla g$

$yz = 2\lambda x$ —— (1)

$xz = 2\lambda y$ —— (2)

$xy = 2\lambda z$ —— (3)

Mult (1) by x, (2) by y, (3) by z & adding

$3xyz = 2\lambda(x^2+y^2+z^2) = 2\lambda$, using (A).

$\div$ (1): $\quad 3x = \frac{1}{x} \quad \therefore x^2 = \frac{1}{3} \quad \therefore x = \pm\frac{1}{\sqrt{3}}$.

$\div$ (2): $\quad 3y = \frac{1}{y} \quad \therefore y^2 = \frac{1}{3} \quad \therefore y = \pm\frac{1}{\sqrt{3}}$.

Similarly $z = \pm\frac{1}{\sqrt{3}}$.

$\therefore$ 8 C.P. are $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

Also, from (1), we can have other cases of C.P.:
$(\lambda \neq 0)$

$x=0, y=0 \quad \therefore z = \pm 1$ (using (A))

(2), $\quad y=0, z=0 \quad \therefore x = \pm 1 \quad$ "

and (3) $\quad z=0, x=0 \quad \therefore y = \pm 1 \quad$ "

$\therefore$ 6 CP are $(0,0,\pm 1), (\pm 1,0,0)$ & $(0,\pm 1,0)$.

Problem 7

(a) $\int_C F = \int_C x^2\,dx + xy\,dy + 1\,dz$

C: $x = t \to dx = 1 \cdot dt$ $\quad 0 \le t \le 1$

$y = t^2 \to dy = 2t\,dt$

$z = 1 \to dz = 0$

$$\therefore \int_C F = \int_0^1 t^2\,dt + t^3 \cdot 2t\,dt + 0$$

$$= \frac{t^3}{3} + \frac{2t^5}{5} \Big]_0^1 = \frac{1}{3} + \frac{2}{5} = \underline{\frac{11}{15}}.$$

(b) $\int_C F = \int \cos z\,dx + e^x\,dy + e^z\,dz$

C: $x = 1 \to dx = 0$ $\quad 0 \le t \le 2$

$y = t \to dy = 1 \cdot dt$

$z = e^t \to dz = e^t\,dt$

$$\therefore \int_C F = \int_0^2 \cos e^t \cdot 0 + e^1\,dt + e^{e^t} e^t\,dt$$

$$= e \cdot t + e^{e^t} \Big]_0^2$$

$\int f' e^f\,dt = e^f.$

$$= 2e + e^{e^2} - e^1$$

$$= \underline{e + e^{e^2}}.$$