Spring '05.

MATH 331 (L 20)

Assignment 4.

Due date : Tutorial Class.

1. Text book, pg 118, prob. 4, 9, 13.

2. pg. 140, 9, 11, 12.

3. Find the maximum/minimum values of
$$f(x,y) = x^2 + 2y^2 - x \text{ on the closed circle } D = x^2 + y^2 \leq 1.$$

4. Determine the extreme values of
$$f(x,y,z) = x + z \text{ subject to the constraint } x^2 + y^2 + z^2 = 1$$

5. Find the max/min values of
$$f(x,y,z) = x^2 + y^2 + 2z^2 \text{ subject to } 2x^2 + y^2 + z^2 = 1.$$

6. Find all critical points of
$$f = xyz \text{ subject to } x^2 + y^2 + z^2 = 1$$
[Hint: 14 critical pts!]

7. Evaluate the curve integral:

(a) $\int_C F$ where $F(x,y,z) = (x^2, xy, 1)$ and
$$c(t) = (t, t^2, 1), \quad 0 \leq t \leq 1$$

(b) $\int_C \cos(z)\,dx + e^x\,dy + e^z\,dz$,
$$\text{where } C(t) = (x(t), y(t), z(t)) = (1, t, e^t), \quad 0 \leq t \leq 2.$$

IV, §6. EXERCISES

(All functions are assumed to be differentiable as needed.)

1. If $x = u(r, s, t)$ and $y = v(r, s, t)$ and $z = f(x, y)$, write out the formula for

$$\frac{\partial z}{\partial r} \quad \text{and} \quad \frac{\partial z}{\partial t}.$$

2. Find the partial derivatives with respect to x, y, s, and t for the following functions.
   (a) $f(x, y, z) = x^3 + 3xyz - y^2z$, $x = 2t + s$, $y = -t - s$, $z = t^2 + s^2$
   (b) $f(x, y) = (x + y)/(1 - xy)$, $x = \sin 2t$, $y = \cos(3t - s)$

3. Let f be a differentiable function on $\mathbf{R}^3$ and suppose that

$$D_1f(0, 0, 0) = 2, \qquad D_2f(0, 0, 0) = D_3f(0, 0, 0) = 3.$$

   Let $g(u, v) = f(u - v, u^2 - 1, 3v - 3)$. Find $D_1g(1, 1)$.

4. Assume that f is a function satisfying

$$f(tx, ty) = t^m f(x, y).$$

   for all numbers x, y, and t. Show that

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x\, \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = m(m - 1)f(x, y).$$

   [*Hint*: Differentiate twice with respect to t. Then put $t = 1$.]

9. Let c be a constant, and let $z = \sin(x + ct) + \cos(2x + 2ct)$. Show that

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}.$$

10. Let c be a constant and let $z = f(x + ct) + g(x - ct)$. Let

$$u = x + ct \quad \text{and} \quad v = x - ct.$$

   Show that

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2} = c^2(f''(u) + g''(v)).$$

11. Let $z = f(u, v)$ and $u = x + y$, $v = x - y$. Show that

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}.$$

12. Let $z = f(x + y) - g(x - y)$. Let $u = x + y$ and $v = x - y$. Show that

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2} = f''(u) - g''(v).$$

13. Let n be a positive integer. For each of the following functions $g(r, \theta)$ show that

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} = 0.$$

   (a) $g(r, \theta) = r^n \cos n\theta$ (b) $g(r, \theta) = r^n \sin n\theta$.

Note. A function $f(x, y) = g(r, \theta)$ which satisfies the condition of this exercise is called **harmonic**, and is important in the theory of wave motions. This exercise gives the basic example of harmonic functions.

. Find the maximum value of $x^2 + xy + y^2 + yz + z^2$ on the sphere of radius 1. [Hint: replacing $x^2 + y^2 + z^2$ by 1 makes the problem simpler.]

3. Let $A = (1, 1, -1)$, $B = (2, 1, 3)$, $C = (2, 0, -1)$. Find the point at which the function

$$f(X) = (X - A)^2 + (X - B)^2 + (X - C)^2$$

reaches its minimum, and find the minimum value.

4. Do Exercise 3 in general, for any three distinct vectors

$$A = (a_1, a_2, a_3), \qquad B = (b_1, b_2, b_3), \qquad C = (c_1, c_2, c_3).$$

5. Find the maximum of the function $3x^2 + 2\sqrt{2}xy + 4y^2$ on the circle of radius 3 in the plane.

6. Find the maximum of the function xyz subject to the constraints

$$x \geqq 0,\ y \geqq 0,\ z \geqq 0, \qquad \text{and} \qquad xy + yz + xz = 2.$$

7. By completing the square show that the only solution of

$$5x^2 + 6xy + 5y^2 = 0$$

is the origin in the plane.

8. Find the extreme values of the function $\cos^2 x + \cos^2 y$ subject to the constraint $x - y = \pi/4$ and $0 \leqq x \leqq \pi$.

9. Find the points on the surface $z^2 - xy = 1$ nearest to the origin.

10. Find the extreme values of the function xy subject to the condition

$$x + y = 1.$$

11. Find the shortest distance between the point $(1, 0)$ and the curve $y^2 = 4x$.

12. Find the maximum and minimum points of the function

$$f(x, y, z) = x + y + z$$

in the region $x^2 + y^2 + z^2 \leqq 1$.

13. Find the extremum values of the function $f(x, y, z) = x - 2y + 2z$ on the sphere $x^2 + y^2 + z^2 = 1$.

14. Find the maximum of the function $f(x, y, z) = x + y + z$ on the sphere

$$x^2 + y^2 + z^2 = 4.$$

15. (a) Find the extreme values of the function f given by $f(x, y, z) = xyz$ subject to the condition $x + y + z = 1$.