Classification of 1st Order O.D.E.

(A) $$\frac{dy}{dx} = f(x,y)$$

(1) Exact Diff. Eq.:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact, if the equality holds:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \qquad \text{(Partial Derivatives)}$$

Short Method of Solution of an exact Diff. Eq.

Ex. $$(y+x)\,dx + (x-y-\sin y+1)\,dy = 0.$$

The eq. is exact, since $\frac{\partial M}{\partial y} = 1$ & $\frac{\partial N}{\partial X} = 1$.

Integrate:

$$\int (y+x)\,\partial x + \int (x-y-\sin y+1)\,\partial y = c$$

$$\therefore\; yx + \frac{x^2}{2} + \cancel{xy} - \frac{y^2}{2} + \cos y + y = c$$

(Delete one of the two common terms: xy).

The general solution is

$$yx + \frac{x^2}{2} - \frac{y^2}{2} + \cos y + y = c$$

(2) Sometimes a non-exact D.E. can become exact by multiplying by a function — Integrating Factor.

Ex. $(4x+3y^2)\,dx + 2xy\,dy = 0$ not exact.

Mult. by Integrating Factor x^2:

$$x^2(4x+3y^2)\,dx + x^2 \cdot 2xy\,dy = 0$$

or $$(4x^3 + 3x^2y^2)\,dx + 2x^3y\,dy = 0$$

This is exact, since $\frac{\partial M}{\partial y} = 6x^2y$, $\frac{\partial N}{\partial x} = 6x^2y$

Integrating,

$$x^4 + x^3y^2 + 2x^3\frac{y^2}{2} = c$$

or $\underline{x^4 + x^3y^2 = c}$ (deleting one common term x^3y^2)

(3) Separable variables (or equation)

If in (A) $\frac{dy}{dx} = f(x,y)$,

the function $f(x,y)$ becomes $g(x)h(y)$ or $\frac{g(x)}{h(y)}$ or $\frac{h(y)}{g(x)}$,

then we have a Diff. Eq., where the two variables x, y are separable.

Ex. $$\frac{dy}{dx} = \frac{(2-x)(3+y)}{y\,e^x}$$

Can be rewritten as:

$$\frac{y}{3+y}\,dy = \frac{2-x}{e^x}\,dx \quad \text{(Separable)}$$

Integrating:

$$\int \frac{y}{3+y}\,dy = \int \frac{2-x}{e^x}\,dx + c \quad (c = \text{arb. const.})$$

<u>You Complete the integrations</u>

(4) Homogeneous Diff. Eq.

A function $f(x,y)$ is homogeneous of degree n, if $f(tx, ty) = t^n f(x,y)$ (i.e. set $x \to tx$, $y \to ty$ + take out t^n)

A function $f(x,y)$ is homo. of degree 0,
if $f(tx, ty) = f(x,y)$.

The Diff. Eq.

(B) $$\frac{dy}{dx} = \frac{g(x,y)}{h(x,y)}$$

is <u>homogeneous</u> if both functions g, h are homogeneous of same degree.

In other words, (B) is homogeneous if the function $\frac{g}{h}$ is homogeneous of degree 0.

Ex.
$$\frac{dy}{dx} = \frac{x^2-y^2}{x^2+2y^2+xy}$$ is homogeneous,

Since $g = x^2-y^2$ and $h = x^2+2y^2+xy$ are both homo. with same degree 2.

On the other hand,

$$\frac{dy}{dx} = \frac{x^2-y}{x^2+y^2}$$ is not homo., (x^2-y is not even homo.)

and

$$\frac{dy}{dx} = \frac{x^2}{x^2-x+y^2+1}$$ is not homo, (denominator is not homo.)

Solve:
$$\frac{dy}{dx} = \frac{y-x}{3x}$$ → both homo. of degree 1; or $\frac{y-x}{3x}$ is homo. of degree 0

It is homogeneous.

Method of Solution

Let $y = ux$ (introduce a new variable $u = u(x)$)
i.e. $u = \frac{y}{x}$

Then

$$\frac{dy}{dx} = x\frac{du}{dx} + u$$

$$\therefore \quad x\frac{du}{dx} + u = \frac{ux - x}{3x} = \frac{x(u-1)}{3x} = \frac{u-1}{3} \quad \text{(no "}x\text{" term)}$$

$$\therefore \quad x\frac{du}{dx} = \frac{u-1}{3} - u = \frac{u-1-3u}{3} = \frac{-1-2u}{3}$$

Separable!

$$\therefore \quad \frac{3\,du}{-1-2u} = \frac{dx}{x}$$

Integrating $\displaystyle\int \frac{3\,du}{-(1+2u)} = \int \frac{dx}{x}$

$$\frac{3}{-2}\ln|1+2u| = \ln x + c$$

Substitute $u = \frac{y}{x}$ to get

$$-\frac{3}{2}\ln\left|1 + \frac{2y}{x}\right| = \ln x + c$$

You can try to simplify

Try to Solve

$$\frac{dy}{dx} = \frac{xy + x^2}{y^2}$$

(5) Linear Eq. (linear in y)

When the diff. eq. can be expressed as

(a) $1.\frac{dy}{dx} + P(x)y = Q(x)$ (where P & Q are fns. of x only)

Then we have a Linear equation.

Method of Solution of (a).

(i) Find the integrating factor,

$$I.F. = e^{\int P(x)dx}$$

(ii) Then the general solution of (a) is

Note: If (a) is $\frac{dy}{dx} - P(x)y = Q(x)$, then $I.F. = e^{-\int P(x)dx}$.

(b) $$e^{\int P(x)dx} \cdot y = \int Q(x) e^{\int P(x)dx} dx + c$$

or

(b) $$y = e^{-\int P(x)dx}\left[\int Q(x) e^{\int P(x)dx} dx + C\right].$$

Why? Go through the example below to see how:

Ex. Solve $\frac{dy}{dx} = (2x-1)y + e^{x^2}$

(a) $\frac{dy}{dx} - (2x-1)y = e^{x^2}$ (Linear as in (a)).

(i) $I.F.\ e^{-\int(2x-1)dx}$

$$= e^{-(x^2-x)} = e^{-x^2+x}$$

Multiplying (a) by e^{-x^2+x}, we get

$$e^{-x^2+x}\left(\frac{dy}{dx} - (2x-1)y\right) = e^{-x^2+x}\left(e^{x^2}\right)$$

The L.H.S. is, exactly (always),

Derivative of a product

$$\frac{d}{dx}\left(e^{-x^2+x}y\right) = e^{-x^2+x}\left(e^{x^2}\right)$$

$$= e^{x} \quad \text{(simplified)}$$

Integrating,

(Same as (6) above)

$$e^{-x^2+x}y = \int e^{x}dx = e^{x}+c$$

$$y = e^{x^2-x}\left(e^{x}+c\right)$$

$$= e^{x^2} + ce^{x^2-x}$$

is the general solution.

You try to solve

Ex. $\dfrac{dy}{dx} - \dfrac{y}{x^2} = 1 - x\sin x \left(\text{or } x^2e^{x}\right)$

(6) Bernoulli Eq. is of the form

(a) $\dfrac{dy}{dx} + P(x)y = Q(x)y^{n}$ (n can be any no.) If $n=0$, then we have a linear eq.

Method of Solution

(i) Divide by y^{n}:

(a_1) $\dfrac{1}{y^{n}}\dfrac{dy}{dx} + \dfrac{P(x)\,y}{y^{n}} = Q(x)$

(a_2) $\dfrac{1}{y^{n}}\dfrac{dy}{dx} + P(x)y^{1-n} = Q(x)$

(ii) Let $u = y^{1-n}$

Then $\frac{du}{dx} = (1-n)y^{1-n-1} \cdot \frac{dy}{dx}$ (Chain rule)

$= (1-n)y^{-n}\frac{dy}{dx}$

$= (1-n)\frac{1}{y^n}\frac{dy}{dx}$

$\therefore \frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$

Hence (a) becomes

$$\frac{1}{1-n}\frac{du}{dx} + P(x)u = Q(x)$$

(b) $\therefore \quad \frac{du}{dx} + \underline{(1-n)P(x)}\,u = (1-n)Q(x)$ (linear eq.)

[Under the transformation $u = y^{1-n}$, (a) $\rightarrow$ (b)]

Recognize that (b) is now a Linear Eq. (in u).

To solve (b), I.F. $= e^{(1-n)\int P(x)dx}$.

Proceed as before for linear eq. (in u).

The general solution is

$$e^{(1-n)\int P(x)dx}\, u = (1-n)\int e^{(1-n)\int P(x)dx} Q(x)dx + C$$

where $u = y^{1-n}$.

Try to solve: $4xy\frac{dy}{dx} = 1 + y^2$

(7) Riccati's Eq.

$$\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x)$$ named after Riccati.

To solve it, one solution is known (or given), say,

$$y = f(x).$$

$\longrightarrow$ Then set $y = f(x) + \frac{1}{u}$.

This reduces Riccati's Eq. to a linear eq. in u.

Ex. $$\frac{dy}{dx} = (1-x)y^2 + (2x-1)y - x.$$

One given solution is $y = f(x) = 1$.

Then let $y = 1 + \frac{1}{u}$ to get

$$\frac{dy}{dx} = -\frac{1}{u^2}\frac{du}{dx} = (1-x)\left(1+\frac{1}{u}\right)^2 + (2x-1)\left(1+\frac{1}{u}\right) - x$$

$$= (1-x)\left(1+\frac{2}{u}+\frac{1}{u^2}\right) + (2x-1)\left(1+\frac{1}{u}\right) - x$$

$$= \frac{1}{u} + \frac{1-x}{u^2}$$ (after cancelling + simplifying)

$$\therefore \frac{du}{dx} = -u^2\left(\frac{1}{u} + \frac{1-x}{u^2}\right)$$

$$= -u + (x-1)$$ linear in u.

$$\therefore \frac{du}{dx} + u = x - 1$$

$$I.F. = e^{\int 1\,dx} = e^x$$

The general solution is $$e^x u = \int e^x(x-1)\,dx$$

$$= e^x(x-1) - \int e^x \cdot 1\,dx$$

$$= e^x(x-2) + c$$

$\frac{1}{u} = y - 1$

$\therefore u = \frac{1}{y-1} \rightarrow$ $\therefore$ $u = (x-2) + ce^{-x}$