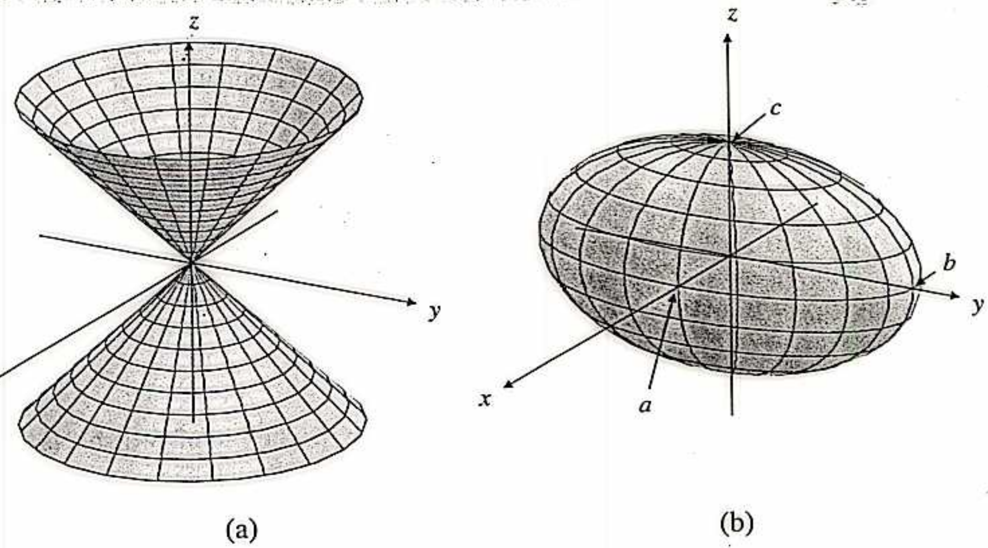


Figure 10.35

(a) The circular cone $a^2z^2 = x^2 + y^2$

(b) The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

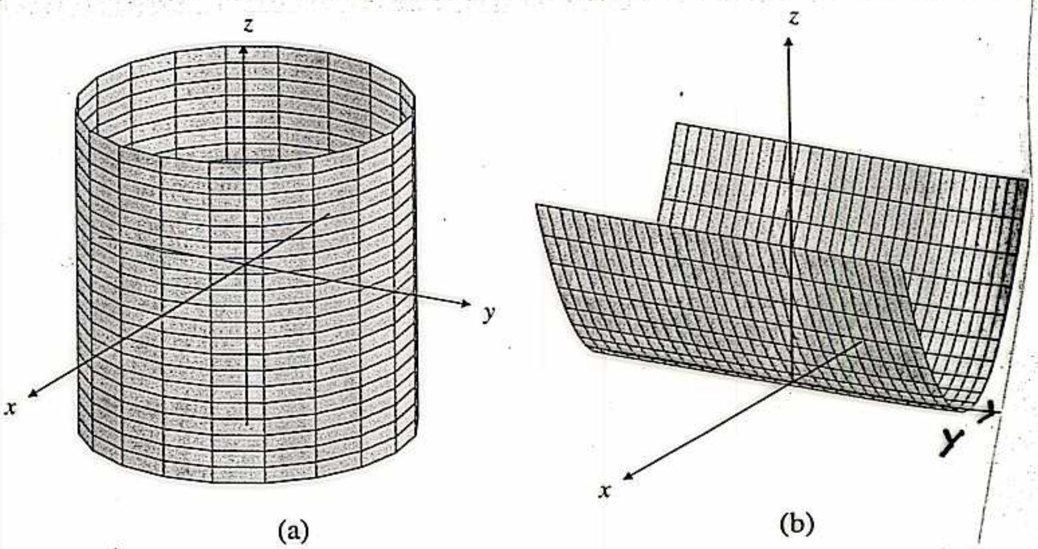


Figure 10.34

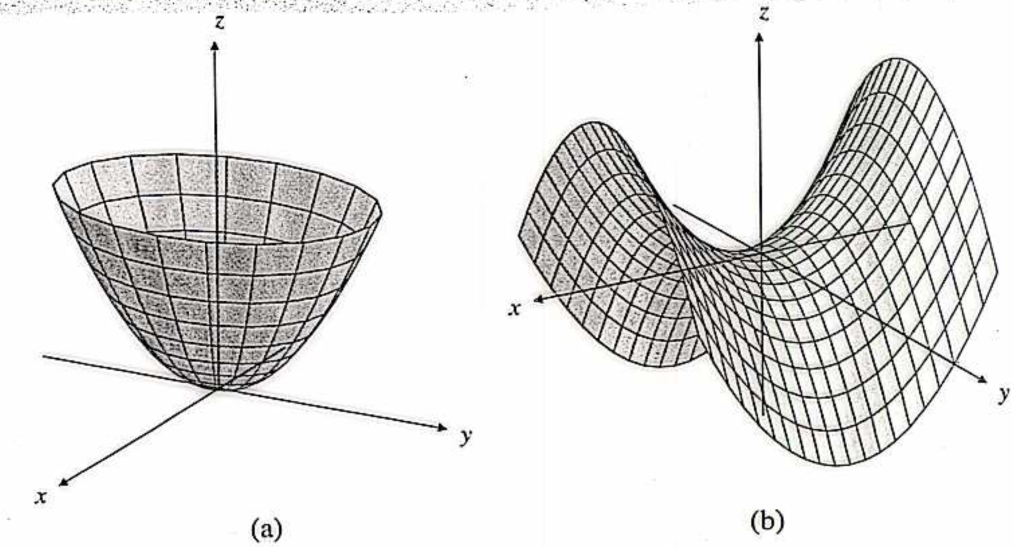
(a) The circular cylinder $x^2 + y^2 = a^2$

(b) The parabolic cylinder $z = x^2$

Figure 10.36

(a) The elliptic paraboloid
$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

(b) The hyperbolic paraboloid
$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$



Hyperboloids. The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

represents a surface called a **hyperboloid of one sheet.** (See Figure 10.37(a).) The equation

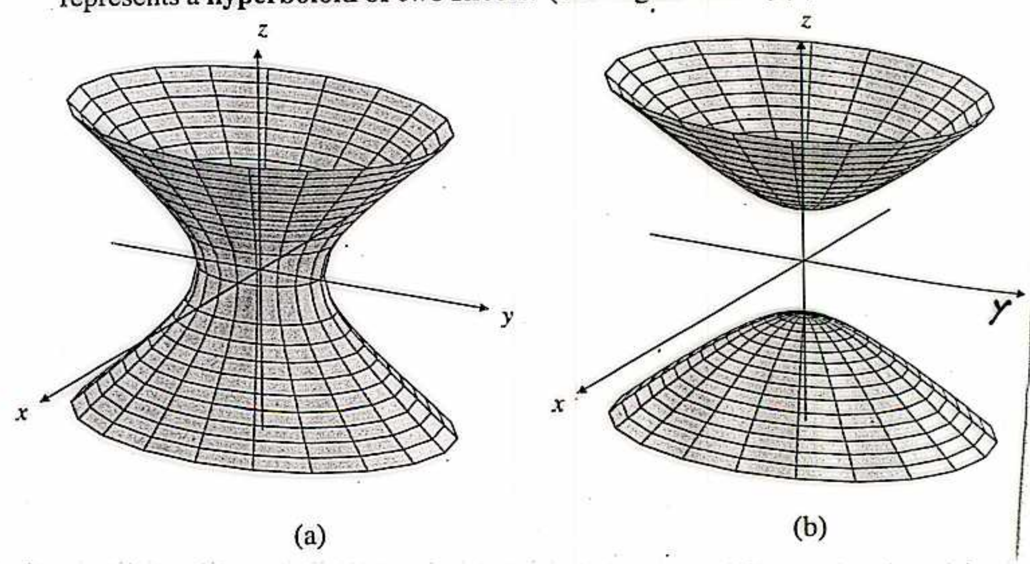
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

represents a **hyperboloid of two sheets.** (See Figure 10.37(b).) Both surfaces

Figure 10.37

(a) The hyperboloid of one sheet
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

(b) The hyperboloid of two sheets
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$



Graphical Representations

The graph of a *function* f of one variable (i.e., the graph of the *equation* $y = f(x)$) is the set of points in the xy-plane having coordinates $(x, f(x))$, where x is in the domain of f. Similarly, the graph of a *function* f of two variables (the graph of the *equation* $z = f(x, y)$) is the set of points in 3-space having coordinates $(x, y, f(x, y))$, where (x, y) belongs to the domain of f. This graph is a surface in $\mathbb{R}^3$ lying above (if $f(x, y) > 0$) or below (if $f(x, y) < 0$) the domain of f in the xy-plane. (See Figure 12.1.) The graph of a function of three variables is a three-dimensional *hypersurface* in 4-space, $\mathbb{R}^4$. In general, the graph of a function of n variables is an n-dimensional *surface* in $\mathbb{R}^{n+1}$. We will not attempt to *draw* graphs of functions of more than two variables!

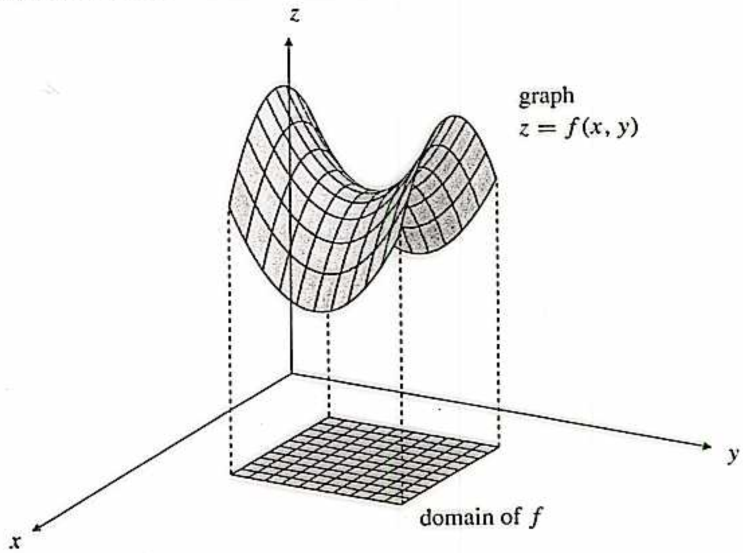


Figure 12.1 The graph of $f(x, y)$ is the surface with equation $z = f(x, y)$ defined for points (x, y) in the domain of f

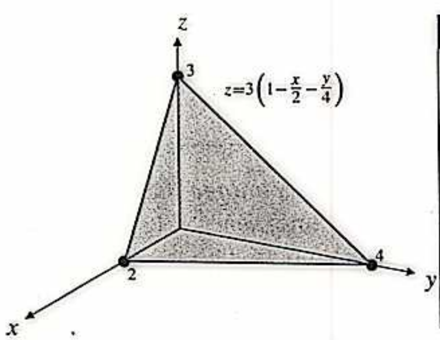


Figure 12.2

Example 1 Consider the function

$$f(x, y) = 3\left(1 - \frac{x}{2} - \frac{y}{4}\right), \qquad (0 \le x \le 2, \quad 0 \le y \le 4 - 2x).$$

The graph of f is the plane triangular surface with vertices at $(2, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 3)$. (See Figure 12.2.) If the domain of f had not been explicitly stated to be a particular set in the xy-plane, the graph would have been the whole plane through these three points.

Example 2 Consider $f(x, y) = \sqrt{9 - x^2 - y^2}$. The expression under the square root cannot be negative, so the domain is the disk $x^2 + y^2 \le 9$ in the xy-plane.

If we square the equation $z = \sqrt{9 - x^2 - y^2}$, we can rewrite the result in the form $x^2 + y^2 + z^2 = 9$. This is a sphere of radius 3 centred at the origin. However, the graph of f is only the upper hemisphere where $z \ge 0$. (See Figure 12.3.)

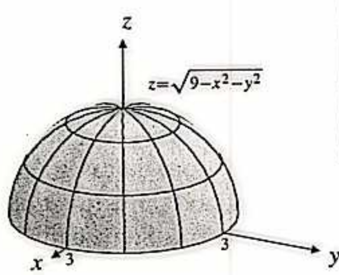


Figure 12.3

Since it is necessary to project the surface $z = f(x, y)$ onto a two-dimensio page, most such graphs are difficult to sketch without considerable artistic tal and training. Nevertheless, you should always try to visualize such a graph sketch it as best you can. Sometimes it is convenient to sketch only part of a gr for instance, the part lying in the first octant. It is also helpful to determine (and sketch) the intersections of the graph with various planes, especially the coordinate planes, and planes parallel to the coordinate planes. (See Figure 12.1.)

Some mathematical software packages will produce plots of three-dimensional graphs to help you get a feeling for how the corresponding functions behave. Figure 12.1 is an example of such a computer-drawn graph, as is Figure 12.4 below. Along with most of the other mathematical graphics in this book, both were produced using the mathematical graphics software package **MG**. Later in this section we discuss how to use Maple to produce such graphs.

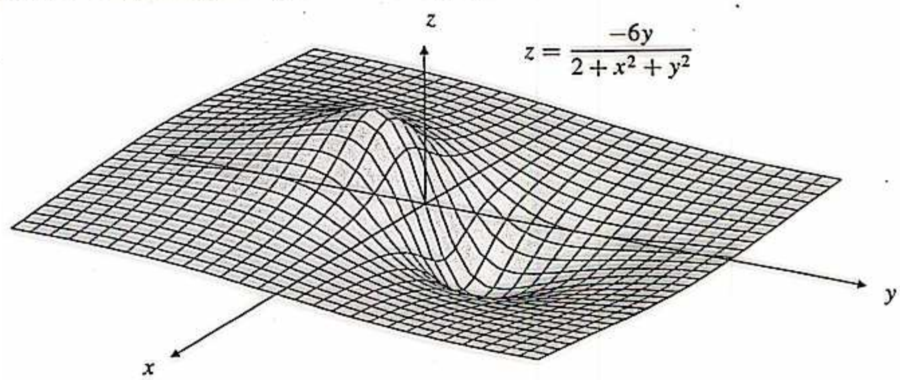


Figure 12.4 The graph of $z = \dfrac{-6y}{2 + x^2 + y^2}$

Another way to represent the function $f(x, y)$ graphically is to produce a two-dimensional *topographic map* of the surface $z = f(x, y)$. In the xy-plane we sketch the curves $f(x, y) = C$ for various values of the constant C. These curves are called **level curves** of f because they are the vertical projections onto the xy-plane of the curves in which the graph $z = f(x, y)$ intersects the horizontal (level) planes $z = C$. The graph and some level curves of the function $f(x, y) = x^2 + y^2$ are shown in Figure 12.5. The graph is a circular paraboloid in 3-space; the level curves are circles centred at the origin in the xy-plane.

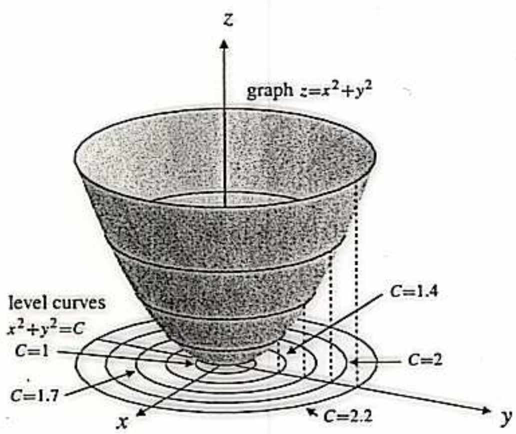


Figure 12.5 The graph of $f(x, y) = x^2 + y^2$ and some level curves of f

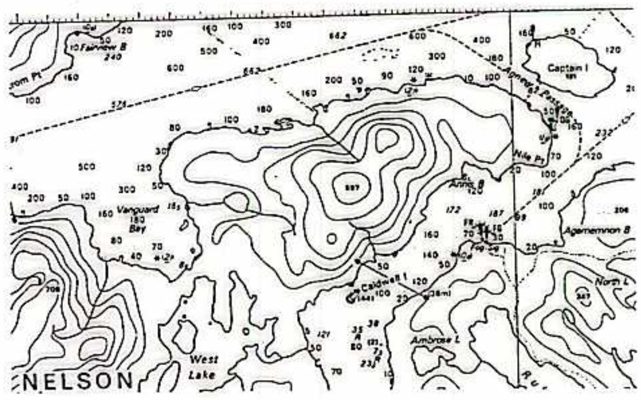
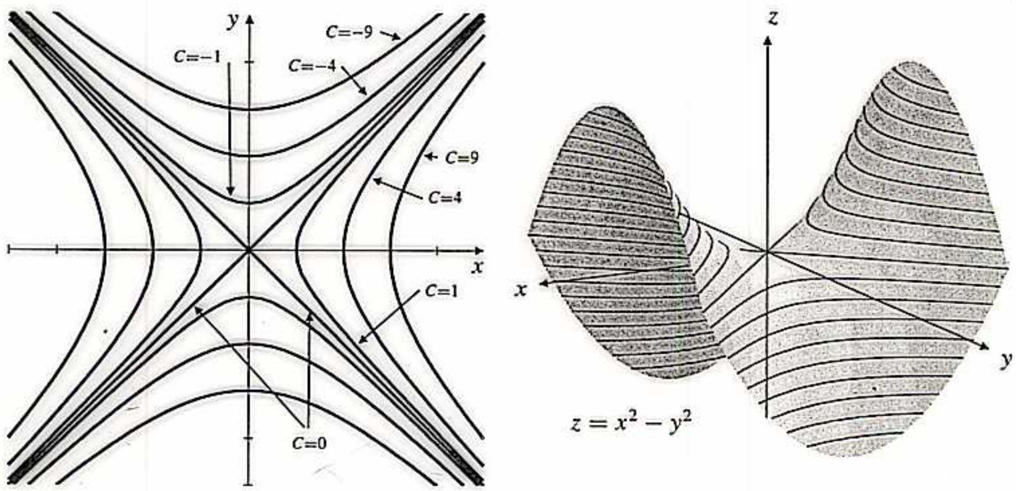


Figure 12.6 Level curves (contours) representing elevation in a topographic map

The contour curves in the topographic map in Figure 12.6 show the elevations, in 100 m increments above sea level, on part of Nelson Island on the British Columbia coast. Since these contours are drawn for equally spaced values of C, the spacing of the contours themselves conveys information about the relative steepness at various places on the mountains; the land is steepest where the contour lines are closest together. Observe also that the streams shown cross the contours at right angles.

.8
l curves of $x^2 - y^2$
graph of $x^2 - y^2$



(a) (b)

Example 6 Describe and sketch some level curves of the function $z = g(x, y)$ defined by $z \geq 0$, $x^2 + (y - z)^2 = 2z^2$. Also sketch the graph of g.

Solution The level curve $z = g(x, y) = C$ (where C is a positive constant) has equation $x^2 + (y - C)^2 = 2C^2$ and is, therefore, a circle of radius $\sqrt{2}C$ centred at $(0, C)$. Level curves for C in increments of 0.1 from 0 to 1 are shown in Figure 12.9(a). These level curves intersect rays from the origin at equal spacing (the spacing is different for different rays) indicating that the surface $z = g(x, y)$ is an oblique circular cone. See Figure 12.9(b).

2.9
el curves of $z = g(x, y)$ for
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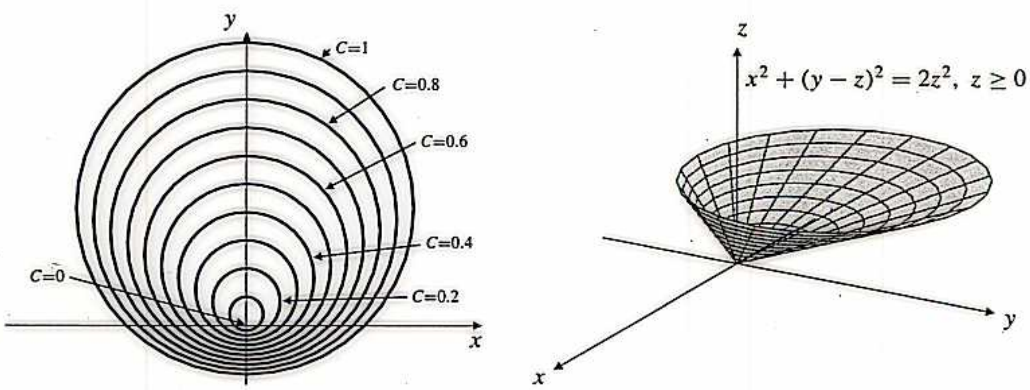
graph of $z = g(x, y)$



(a) (b)

Although the *graph* of a function $f(x, y, z)$ of three variables cannot easily be drawn (it is a three-dimensional *hypersurface* in 4-space), such a function has **level surfaces** in 3-space that can, perhaps, be drawn. These level surfaces have equations $f(x, y, z) = C$ for various choices of the constant C. For instance, the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$ are concentric spheres centred at the origin. Figure 12.10 shows a few level surfaces of the function $f(x, y, z) = x^2 - z$. They are parabolic cylinders.

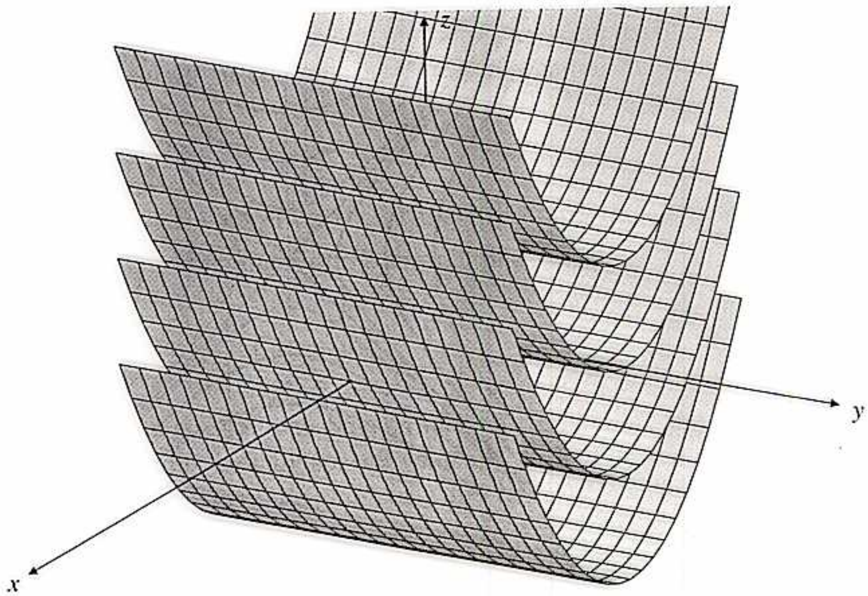


Figure 12.10 Level surfaces of $f(x, y, z) = x^2 - z$

Using Maple Graphics

Like many mathematical software packages, Maple has several plotting routines to help you visualize the behaviour of functions of two and three variables. We mention only a few of them here; there are many more. Most of the plotting routines are in the **plots** package, so you should begin any Maple session where you want to use them with the input

```
> with(plots):
```

To save space, we won't show any of the plot output here. You will need to play with modifications to the various plot commands to obtain the kind of output you desire.

The graph of a function $f(x, y)$ of two variables (or an expression in x and y) can be plotted over a rectangle in the xy-plane with a call to the **plot3d** routine. For example,

```
> f := -6*y/(2+x^2+y^2);
> plot3d(f, x=-6..6, y=-6..6);
```

will plot a surface similar to the one in Figure 12.4 but without axes and viewed from a steeper angle. You can add many kinds of options to the command to change the output. For instance,

```
> plot3d(f, x=-6..6, y=-6..6, axes=boxed,
    orientation=[30,70]);
```

will plot the same surface within a 3-dimensional rectangular box with scales on three of its edges indicating the coordinate values. (If we had said `axes=normal` instead, we would have got the usual coordinate axes through the origin, but they tend to be harder to see against the background of the surface, so `axes=boxed` is usually preferable. The option `orientation=[30,70]` results in the plot's being viewed from the direction making angle 70° with the z-axis and lying in a plane containing the z-axis making an angle 30° with the xz-plane. (The default value of the orientation is [45, 45] if the option is not specified.) By default, the surface plotted by plot3d is ruled by two families of curves, representing its intersection with vertical planes $x = a$ and $y = b$ for several equally spaced values of a and b, and it is coloured opaquely so that hidden parts do not show.

Instead of plot3d, you can use **contourplot3d** to get a plot of the surface ruled by contours on which the value of the function is constant. If you don't get enough contours by default, you can include a `contours=n` option to specify the number you want.

```
> contourplot3d(f, x=-6..6, y=-6..6, axes=boxed,
    contours=24);
```

The contours are the projections of the level curves onto the graph of the surface. Alternatively, you can get a two-dimensional plot of the level curves themselves using **contourplot**

```
> contourplot(f, x=-6..6, y=-6..6, axes=normal,
    contours=24);
```

Other options you may want to include with plot3d or contourplot3d are:

(a) `view=zmin..zmax` to specify the range of values of the function (i.e., z) to show in the plot.

(b) `grid=[m,n]` to specify the number of x and y values at which to evaluate the function. If your plot doesn't look smooth enough, try $m = n = 20$ or 30 or even higher values.

The graph of an equation, $f(x, y) = 0$, in the xy-plane can be generated without solving the equation for x or y first, by using **implicitplot**.

```
> implicitplot(x^3-y^2-5*x*y-x-5, x=-6..7, y=-5..6);
```

will produce the graph of $x^3 - y^2 - 5xy - x - 5 = 0$ on the rectangle $-6 \le x \le 7$, $-5 \le y \le 6$. There is also an **implicitplot3d** routine to plot the surface in 3-space having an equation of the form $f(x, y, z) = 0$. For this routine you must specify ranges for all three variables;

```
> implicitplot3d(x^2+y^2-z^2-1, x=-4..4, y=-4..4,
    z=-3..3, axes=boxed);
```

plots the hyperboloid $z^2 = x^2 + y^2 - 1$.

Finally, we observe that Maple is no more capable than we are of drawing graphs of functions of three or more variables, since it doesn't have four-dimensional plot capability. The best we can do is plot a set of level surfaces for such a function:

```
> implicitplot3d({z-x^2-2,z-x^2,z-x^2+2},x=-2..2,
    y=-2..2, z=-2..5, axes=boxed);
```

It is possible to construct a sequence of *plot structures* and assign them to, say, the elements of a list variable, without actually plotting them. Then all the plots can be plotted simultaneously using the **display** function.

```
> for c from -1 to 1 do
    p[c] := implicitplot3d(z^2-x^2-y^2-2*c, x=-3..3,
    y=-3..3, z=0..2,
    color=COLOR(RGB, (1+c)/2, (1-c)/2,1)) od:
> display([seq(p[c],c=-1..1)], axes=boxed,
    orientation=[30,40]);
```

Note that the command creating the plots is terminated with a colon ":" rather than the usual semicolon. If you don't suppress the output in this way, you will get vast amounts of meaningless numerical output as the plots are constructed. The `color=...` option is an attempt to give the three plots a different colour so they can be distinguished from each other.

Example 1 The function $f(x, y) = x^2 + y^2$ (see Figure 13.1) has a critical point at $(0, 0)$ since $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and both components of ∇f vanish at $(0, 0)$. Since

$$f(x, y) > 0 = f(0, 0) \quad \text{if} \quad (x, y) \neq (0, 0),$$

f must have (absolute) minimum value 0 at that point. If the domain of f is not restricted, f has no maximum value. Similarly, $g(x, y) = 1 - x^2 - y^2$ has (absolute) maximum value 1 at its critical point $(0, 0)$. (See Figure 13.2.) ■

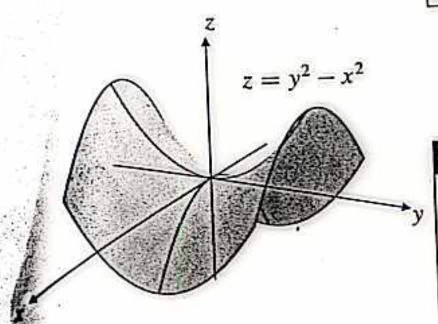


Figure 13.3 $y^2 - x^2$ has a saddle point at $(0, 0)$

Example 2 The function $h(x, y) = y^2 - x^2$ also has a critical point at $(0, 0)$ but has neither a local maximum nor a local minimum value at that point. Observe that $h(0, 0) = 0$ but $h(x, 0) < 0$ and $h(0, y) > 0$ for all nonzero values of x and y. (See Figure 13.3.) The graph of h is a hyperbolic paraboloid. In view of its shape we call the critical point $(0, 0)$ a **saddle point** of h. ■

In general, we will somewhat loosely call any *interior critical* point of the domain of a function f of several variables a **saddle point** if f does not have a local maximum or minimum value there. Even for functions of two variables, the graph will not always look like a saddle near a saddle point. For instance, the function $f(x, y) = -x^3$ has a whole line of *saddle points* along the y-axis (see Figure 13.4), although its graph does not resemble a saddle anywhere. These points resemble inflection points of a function of one variable. Saddle points are higher-dimensional analogues of such horizontal inflection points.

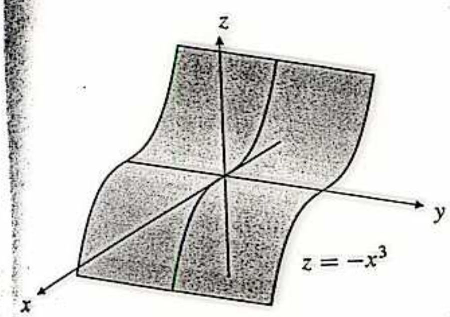


Figure 13.4 A line of saddle points

Example 3 The function $f(x, y) = \sqrt{x^2 + y^2}$ has no critical points but does have a singular point at $(0, 0)$ where it has a local (and absolute) minimum value, zero. The graph of f is a circular cone. (See Figure 13.5(a).) ■

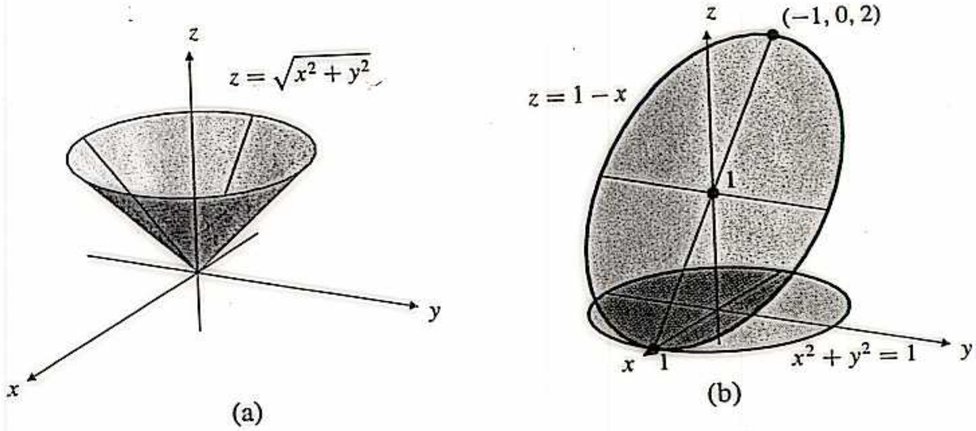


Figure 13.5

(a) $\sqrt{x^2 + y^2}$ has a minimum value at the singular point $(0, 0)$

(b) When restricted to the disk $x^2 + y^2 \leq 1$, the function $1 - x$ has maximum and minimum values at boundary points