Curve Integral

(A)

Let $C(t)$ be a curve in an open set U.
Let F be a vector (field) on U, say,

$$F(x,y) = (f(x,y),\ g(x,y)) \text{ in 2-D, or}$$

$$F(X) = F(x,y,z) = (f(X),\ g(X),\ h(X)) \text{ in 3-D.}$$

Let $C(t)$ be ctsly diffble (i.e. $\frac{dC}{dt}$ exists and is cts).

Defn Let $C(t)$ be defined on $[t_0, t_1]$.
We define the curve integral of $F(X)$ along the curve $C(t)$ as:

$$\int_C F = \int_{t_0}^{t_1} F(C(t)) \cdot \frac{dC}{dt}\, dt$$

or

$$= \int_{t_0}^{t_1} F(C(t)) \cdot dC$$

$$= \int_{t_0}^{t_1} f\,dx + g\,dy + h\,dz.$$

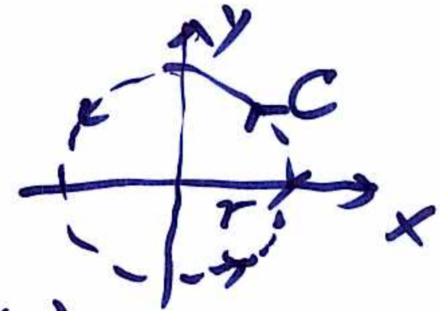
(B1)

Note: $dx = \frac{dx}{dt}dt$, $dy = \frac{dy}{dt}dt$, $dz = \frac{dz}{dt}dt$.
where t is a parameter.

Ex. 1. Let the curve be $C: x^2+y^2=r^2$
In terms of parameter t, this circle is

$$x = r\cos t, \quad y = r\sin t, \qquad 0 \le t \le 2\pi.$$

Let $F(x,y) = (x^2, xy^2)$.



The curve integral is

$$\int_C F = \int (x^2, xy^2)\cdot(dx, dy)$$

$$= \int_C x^2\,dx + xy^2\,dy \qquad \begin{aligned} dx &= -r\sin t\,dt \\ dy &= r\cos t\,dt \end{aligned}$$

$$= \int_0^{2\pi} r^2\cos^2 t\,(-r\sin t\,dt) + r^3\cos t\sin^2 t\,(r\cos t\,dt).$$

$$= \int_0^{2\pi} -r^3\cos^2 t\,\sin t + r^4\cos^2 t\,\sin^2 t\; dt.$$

use $\int f' f^n\, dt = \frac{f^{n+1}}{n+1}$

$$= r^3 \frac{\cos^3 t}{3}\Big]_0^{2\pi} + r^4 \int_0^{2\pi} \left(\tfrac{1}{2}\sin 2t\right)^2 dt.$$

$$= 0 + \frac{r^4}{4}\int_0^{2\pi} \sin^2 2t\; dt$$

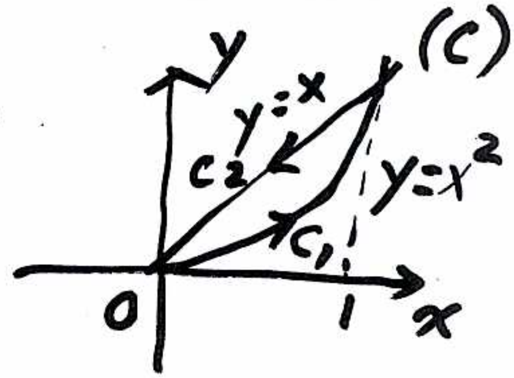
$$= \frac{r^4}{4}\cdot\frac{1}{2}\int_0^{2\pi}(1-\cos 4t)\,dt$$

$$= \frac{r^4}{8}\left[t - \frac{1}{4}\sin 4t\right]_0^{2\pi}$$

$$= \frac{r^4\cdot 2\pi}{8} = \frac{\pi r^4}{4}.$$

The curve integral can be viewed as the work done to move $F(x,y)$ along the curve $C(t)$.

Ex. 2 Let $F(x,y) = (x^2, xy^2)$.
& let the path be
$C_1 : y = x^2$ and $C_2 : y = x$
(as indicated).



$$\int_{C=C_1+C_2} F = \int_{C_1} F + \int_{C_2} F$$

$$= \int_{C_1} (x^2, xy^2)\cdot(dx, dy) + \int_{C_2} (x^2, xy^2)\cdot(dx, dy)$$

Method 1

You can parametrize C by means of t:

$$x = t, \quad y = t^2. \qquad \text{(try yourself)}$$

Method 2

$$\int_C F = \int_{C_1} F + \int_{C_2} F$$

For C_1: $y = x^2 \therefore dy = 2x\,dx$

(D).

Hence $\int_{C_1} F = \int_{C_1} x^2\,dx + xy^2\,dy$

$= \int_0^1 x^2\,dx + x^5 \cdot 2x\,dx$

$= \frac{x^3}{3} + \frac{2x^7}{7}\Big]_0^1 = \frac{1}{3} + \frac{2}{7}$

For C_2: $y = x \;\therefore dy = dx$

$\int_{C_2} F = \int_{C_2} x^2\,dx + xy^2\,dy$

$= \int_1^0 x^2\,dx + x^3\,dx$

$= \frac{x^3}{3} + \frac{x^4}{4}\Big]_1^0 = -\frac{1}{3} - \frac{1}{4}.$

Adding

$\int_{C_1} F + \int_{C_2} F = \frac{2}{7} - \frac{1}{4} = \frac{8-7}{28} = \frac{1}{28}.$

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14. Let again

$$F(x, y) = \left(\frac{xe^r}{r}, \frac{ye^r}{r}\right).$$

(E)

p. 227

Find the value of the integral of this vector field:

(a) From $(2, 1)$ to $(-3, 4)$ along any path not passing through the origin.
(b) From $(2, 0)$ to $(0, 2)$ along the circle of radius 2.
(c) From $(2, 0)$ to $(\sqrt{2}, \sqrt{2})$ along the circle of radius 2.
(d) All the way around the circle of radius 2.

15. Find the integral of the vector field

$$G(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right):$$

(a) Along the line $x + y = 1$ from $(0, 1)$ to $(1, 0)$.
(b) From the point $(2, 0)$ to the point $(-1, \sqrt{3})$ along the path shown on the figure.

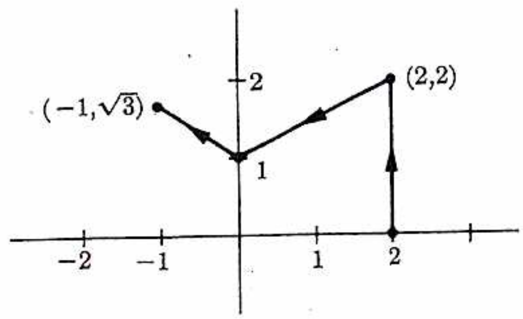

Figure 14

16. Find the integral of the vector field $(x, y^2, 4z^3)$ along the path shown on the figure, from the point $(0, 0, 0)$ to the point $(1, 1, 2)$.

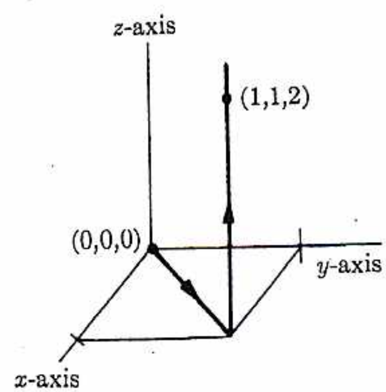

Figure 15