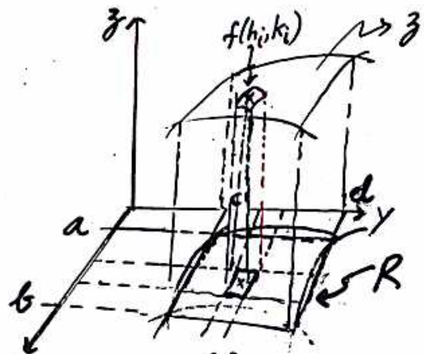
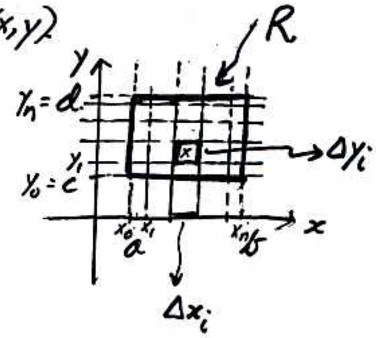
Double Integral

Let the fn $z = f(x,y)$ be defined on a domain (rectangle)
$R:\ a \le x \le b,\ \ c \le y \le d$.



(i)



(ii)

We partition the rectangle R into smaller rectangles by vertical lines $a = x_0 < x_1 < x_2 \cdots < x_n = b$ and horizontal lines $c = y_0 < y_1 < \cdots < y_n = d$. Take a typical subrectangle (as shown in fig. (ii)) with area $\Delta A_i = \Delta x_i \cdot \Delta y_i$. Consider an arbitrary pt. (h_i, k_i) in this subrectangle, and the corresponding value of $f(h_i, k_i)$. If we regard $f(h_i, k_i)$ as the average value of f in the typical subrectangle, then

$$f(h_i, k_i)\,\Delta A_i$$

will approximate the volume of the rectangular solid. Adding up all such 'infinitesimal' volumes, we see that the sum

$$\sum_{i=1}^{n} f(h_i, k_i)\,\Delta x_i\,\Delta y_i$$

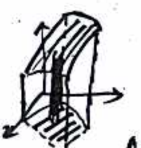
approximates the volume of the entire rectangle R. Obviously, this sum depends on how the rectangle is subdivided. If there are many subrectangles (i.e. $n \to \infty$ and $\Delta x_i \Delta y_i \to 0$), and if the above sum remains the same, then we can say that $\lim\limits_{\Delta x_i \Delta y_i \to 0} \sum\limits_{i=1}^{n} f(h_i, k_i)\,\Delta x_i \Delta y_i$ exists.

This limit is called the Double integral of f on the region R, written as.

$$\lim_{\Delta x_i \Delta y_i \to 0} \sum_{i=1}^{n} f(h_i, k_i)\,\Delta x_i\, \Delta y_i = \iint_R f(x,y)\,dx\,dy$$

<u>Theorem</u> If $f(x,y)$ is a continuous fn on a closed rectangular region R, then the double integral of f on R

$$V = \iint_R f(x,y)\,dx\,dy$$



exists. If $f(x,y) \geq 0$ for all (x,y) in R, then V is the volume of the solid with base R and height f.

<u>Note</u>: The double integral also exists, for any closed region R.

(I) <u>Evaluation of Double Integral when R is a rectangle</u>

To evaluate the double integral, (other than treating it as a limit, which is quite cumbersome) we integrate it by means of <u>iterated integrals</u>:

$$\iint_R f(x,y)\,dx\,dy = \int_{y=c}^{d} \left[\int_{x=a}^{b} f(x,y)\,dx \right] dy \qquad (A)$$

$$\text{or} = \int_{x=a}^{b} \left[\int_{y=c}^{d} f(x,y)\,dy \right] dx \qquad (B)$$

Each integral on the right, $\int_{x=a}^{b} f(x,y)\,dx$ or $\int_{y=c}^{d} f(x,y)\,dy$ is called an <u>iterated integral</u>. You can choose either (A) or (B).

In the first situation (A), we first compute the iterated integral

$$\int_{x=a}^{b} f(x,y)\,dx,$$

regarding y as a constant, and follow by an integration with respect to y.

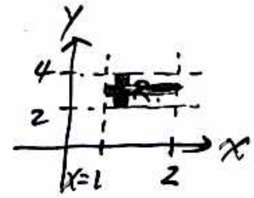
In situation (B), we first compute the iterated integral

$$\int_{y=c}^{d} f(x,y)\,dy,$$

regarding x as a constant, and follow by an integration with respect to x.

Ex. 1. Evaluate

$$I = \int_{y=2}^{4}\int_{x=1}^{2} (3x^2-2y)\,dx\,dy.$$



R is a rectangle

By (A),

$$I = \int_{y=2}^{4}\left[\int_{x=1}^{2}(3x^2-2y)\,dx\right]dy$$

$$= \int_{y=2}^{4}\left[\cancel{3}\frac{x^3}{\cancel{3}}-2yx\right]_1^2 dy$$

$$= \int_2^4 \left[8-4y-(1-2y)\right]dy$$

$$= \int_2^4 7-2y\,dy = 7y-y^2\Big|_2^4 = 28-16-(14-4)$$

$$= 2.$$

or By (B)

$$I = \int_{x=1}^{2}\left[\int_{y=2}^{4}(3x^2-2y)\,dy\right]dx$$

$$= \int_{x=1}^{2}\left[3x^2y-y^2\right]_2^4 dx$$

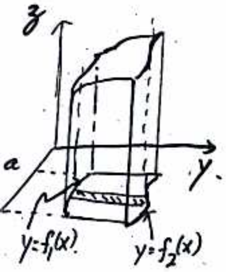
$$= \int_1^2 \left[12x^2-16-(6x^2-4)\right]dx$$

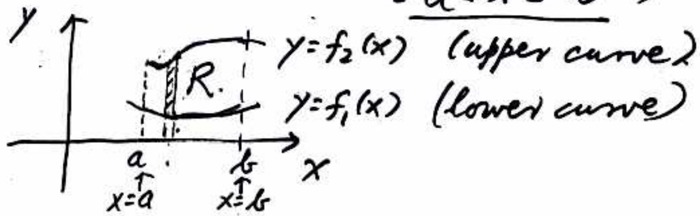
$$= \int_1^2 \left[6x^2-12\right]dx = \left[\frac{6x^3}{3}-12x\right]_1^2 = 16-24-(2-12)$$

$$= 2, \text{ as before.}$$

(II) Evaluation of Double Integral when R is not a rectangle.

(a) R is given (or bounded) by $\begin{cases} f_1(x) \le y \le f_2(x), \\ a \le x \le b \end{cases}$.





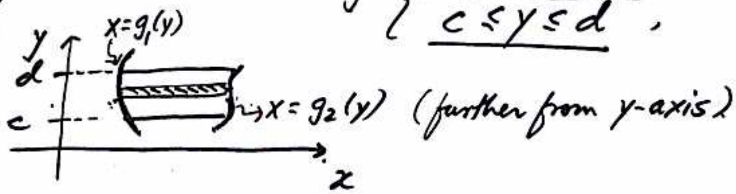
Then we must compute

$$\longrightarrow \quad \iint_R f(x,y)\,dx\,dy = \int_{x=a}^{b} \left[\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy \right] dx$$

(regard x as const.)

(b) R is given (or bounded) by $\begin{cases} g_1(y) \le x \le g_2(y), \\ c \le y \le d \end{cases}$.

$= f_1(x)$
$= g_1(y)$

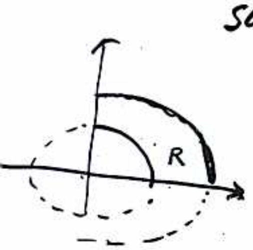


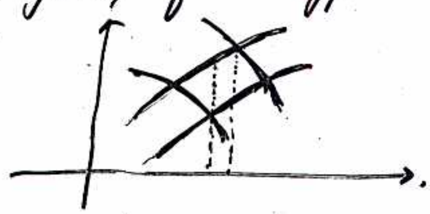
Then we must compute

$$\longrightarrow \quad \iint_R f(x,y)\,dx\,dy = \int_{y=c}^{d} \left[\int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx \right] dy$$

(regard y as const.)

(c) Any other region R should be split up into subregions of the type in (a) or (b).

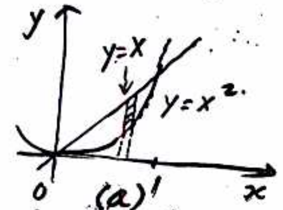


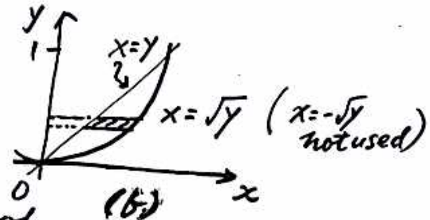


How would you split up these regions?

Ex. 1 Evaluate $\iint_R (x+y-2)\,dx\,dy$ where R is the region bounded by the line $y=x$ and the parabola $y=x^2$

Sketch the region R





First, find the intersection points of

and $\begin{matrix} y=x \\ y=x^2. \end{matrix}$

i.e. $x-x^2=0 \Rightarrow x=0,\ x=1$

(or $y=0,\ y=1$)

Then the double integral can be evaluated by regarding R as of type (a) or as of type (b).

Either (a).

$$\iint_R (x+y-2)\,dx\,dy = \int_{x=0}^{1}\left[\int_{y=x^2}^{x} x+y-2\,dy\right]dx$$

$$= \int_0^1 xy+\frac{y^2}{2}-2y\Big|_{x^2}^{x}\,dx$$

$$= \int_0^1 x^2+\frac{x^2}{2}-2x-\left(x^3+\frac{x^4}{2}-2x^2\right)\,dx$$

$$= \int_0^1 \frac{7}{2}x^2-2x-x^3-\frac{x^4}{2}\,dx$$

$$= \frac{7}{2\cdot 3}x^3-x^2-\frac{x^4}{4}-\frac{x^5}{10}\Big|_0^1 = \frac{7}{6}-1-\frac{1}{4}-\frac{1}{10} = -\frac{11}{60}.$$

Or (b)

$$\iint_R (x+y-2)\,dx\,dy = \int_{y=0}^{1}\left[\int_{x=y}^{\sqrt{y}} x+y-2\,dx\right]dy$$

$$= \int_0^1 \frac{x^2}{2}+xy-2x\Big|_{y}^{\sqrt{y}}\,dy$$

$$= \int_0^1 \frac{y}{2}+y^{3/2}-2y^{1/2}-\left(\frac{y^2}{2}+y^2-2y\right)dy$$

$$= \int_0^1 \frac{5}{2}y+y^{3/2}-2y^{1/2}-\frac{3}{2}y^2\,dx$$

$$= \frac{5}{4}y^2+\frac{2y^{5/2}}{5}-\frac{4y^{3/2}}{3}-\frac{1}{2}y^3\Big|_0^1 = \frac{5}{4}+\frac{2}{5}-\frac{4}{3}-\frac{1}{2} = \frac{-11}{60}.$$

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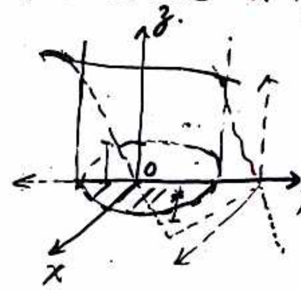
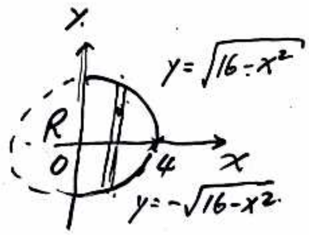
Ex. 2. Find the volume of the solid under the plane $z = 4x$ and above the circle $x^2+y^2=16$ in the xy-plane.

Sketch the solid.





≥ 0

$x \geq 0$

$\geq 0.$

$$\text{Volume} = \iint_R z \, dx\, dy. \quad \text{where } R \text{ is the circle } x^2+y^2=16$$

$(-4 \leq x \leq 0$ is not allowed, since $z \leq 0)$

$$= \int_{x=0}^{4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 4x \, dy. \, dx$$

$$= \int_0^4 4xy \Big|_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} dx$$

$$= 8\int_0^4 x\sqrt{16-x^2}\, dx$$

or let

$u = 16 - x^2$

$du = -2x\,dx$

$$= 8 \cdot -\frac{1}{3}\left(16-x^2\right)^{3/2}\Big|_0^4$$

$-\frac{1}{2}\int \sqrt{u}\, du$

$-\frac{1}{2}\frac{u^{3/2}}{\frac{3}{2}}$

$$= \frac{-8}{3}\left(0 - 16^{3/2}\right)$$

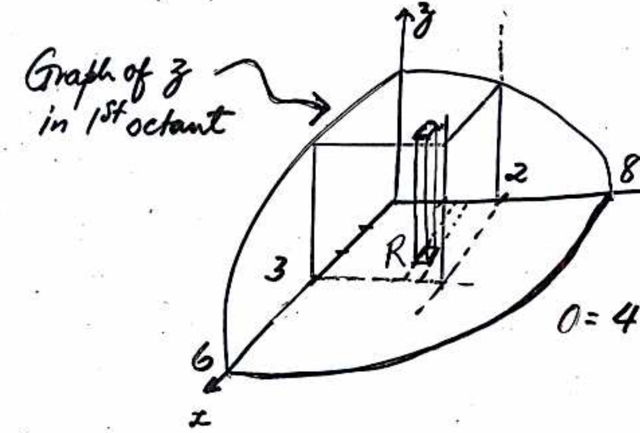
$$= \frac{8 \times 64}{3} = \frac{512}{3} \text{ cu. units.}$$

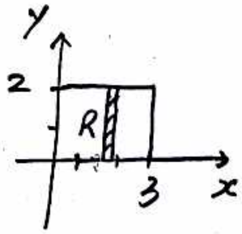
Try:

Ex. Sketch the region enclosed by
$x \geq 0$, $x^2+y^2 \geq 1$, $x^2+y^2 \leq 9$ and $y \leq x$.

Ex. Evaluate $\int_0^4 \int_{\sqrt{x}}^2 \sin \pi y^3 \, dy\, dx$
by drawing the region of integration and reversing the order of integration.

Pg. 108 **Ex. 3.** Find the vol. of the solid bounded by the surface $z = 4 - \frac{x^2}{9} - \frac{y^2}{16}$, the planes $x = 3$, $y = 2$ and the 3 coordinate planes.





$0 = 4 - \frac{x^2}{9} - \frac{y^2}{16}$

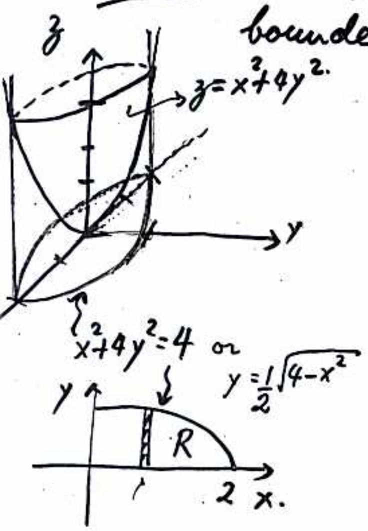
$$\text{Vol.} = \iint_R 4 - \frac{x^2}{9} - \frac{y^2}{16}\, dy\, dx$$

$$= \int_0^3 \int_0^2 4 - \frac{x^2}{9} - \frac{y^2}{16}\, dy\, dx$$

$$= \int_0^3 4y - \frac{x^2 y}{9} - \frac{y^3}{48} \Big|_0^2 dx$$

$$= \int_0^3 \frac{47}{6} - \frac{2}{9}x^2\, dx = \frac{47}{6}x - \frac{2}{27}x^3 \Big|_0^3 = \underline{\underline{21\tfrac{1}{2}}}.$$

Ex. 4 Find the vol. of the solid above the xy-plane bounded by $z = x^2 + 4y^2$ and the cylinder $x^2 + 4y^2 = 4$.



$$\text{Vol.} = 4 \iint_R x^2 + 4y^2\, dy\, dx$$

$$= 4 \int_0^2 \int_0^{\frac{1}{2}\sqrt{4-x^2}} x^2 + 4y^2\, dy\, dx$$

$$= 4 \int_0^2 x^2 y + 4\frac{y^3}{3} \Big|_0^{\frac{1}{2}\sqrt{4-x^2}} dx$$

$$= \frac{4}{3} \int_0^2 (x^2 + 2)\sqrt{4 - x^2}\, dx = \underline{\underline{4\pi}}$$

(Use Subst. $x = 2\sin\theta$!).

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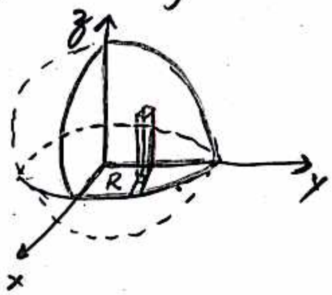
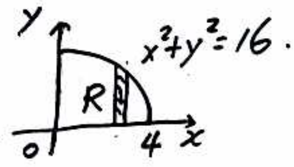
Ex. 5. Find by double integration the volume of the solid bounded by the sphere $x^2+y^2+z^2=16$ and lying in the first ortant (i.e. $x>0, y>0, z>0$).





Soln From the eq of the sphere, at $z=0$, we get

$$x^2+y^2=16.$$

We are interested only in the first octant i.e. in the region R, as shown.

$$R: \begin{cases} 0 \le y \le \sqrt{16-x^2} \\ 0 \le x \le 4 \end{cases}$$

$16-x^2-y^2$

$$\therefore \text{Volume} = \iint_R z \, dy\,dx$$

$$= \int_0^4 \int_0^{\sqrt{16-x^2}} \sqrt{16-x^2-y^2} \, dy.dx$$

Use Subst.

$y = \sqrt{16-x^2}\sin\theta$.

$dy = \sqrt{16-x^2}\cos\theta d\theta$.

also

$\sqrt{16-x^2-y^2} = \sqrt{16-x^2}\cos\theta$;

when $y=0 \Rightarrow \theta=0$

when $y=\sqrt{16-x^2} \Rightarrow \theta=\frac{\pi}{2}$.

$$\longrightarrow = \int_0^4 \int_0^{\pi/2} \sqrt{16-x^2}\cos\theta \sqrt{16-x^2}\cos\theta d\theta . dx$$

$$= \int_0^4 \int_0^{\pi/2} (16-x^2)\cos^2\theta d\theta \; dx$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$= \int_0^4 (16-x^2)\frac{1}{2}\int_0^{\pi/2} (1+\cos 2\theta) d\theta \; dx$$

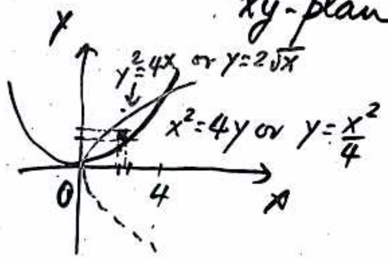
$$= \frac{1}{2}\int_0^4 (16-x^2)\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} dx$$

$$= \frac{1}{2}\int_0^4 (16-x^2)\frac{\pi}{2} \, dx = \frac{\pi}{4}\left[16x - \frac{x^3}{3}\right]_0^4$$

$$= \frac{32}{3}\pi.$$

Pg. 110.

<u>Ex. 6</u> Find the <u>area</u> of the region bounded by the curves $y^2 = 4x$ and $x^2 = 4y$ in the xy-plane.



$$\begin{cases} y^2 = 4x \\ x^2 = 4y \end{cases}$$

$x = 0, 4$

$$A = \iint_R dy\,dx. \longleftarrow$$

$$= \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy\,dx$$

$$= \int_0^4 y \Big|_{\frac{x^2}{4}}^{2\sqrt{x}} dx$$

$$= \int_0^4 2\sqrt{x} - \frac{x^2}{4}\,dx$$

$$= \frac{16}{3}.$$

<u>Change of Variables in Multiple Integration</u>

In double or triple integration, it is sometimes necessary to employ a change of variables (from the original xy coordinates to uv coordinates). This can arise either because of the region of integration or because of the integrand. The change of variables is required to simplify the integration.

Suppose we make the change of variables

$$x = f(u, v), \quad y = g(u, v).$$

This can be interpreted as transforming a region R of the xy-plane into a region S of the uv-plane.

Then we have the following Result:

(A) $$\iint_R F(x,y)\,dx\,dy = \iint_S F[f(u,v), g(u,v)]\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv,$$

where the term $\frac{\partial(x,y)}{\partial(u,v)}$ is called the "Jacobian" of the change of variable (or transformation) defined as

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

Note that in the above integral, $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ is used, rather than $\frac{\partial(x,y)}{\partial(u,v)}$. absolute value

Similarly, in triple integration

$$\iiint_V F(x,y,z)\,dx\,dy\,dz,$$

if, we use the change of variables:

$$x = f(u,v,w),\ y = g(u,v,w),\ z = h(u,v,w)$$

and the region V becomes the region W of the uvw-space, then we have this Result:

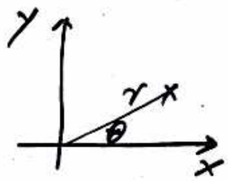
(B) $$\iiint_V F(x,y,z)\,dx\,dy\,dz = \iiint_W F(f,g,h)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw,$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Ex. 1 From xy-coor. to $r\theta$-polar coor., namely,

(1) $$\begin{cases} x = r\cos\theta \\ y = r\sin\theta . \end{cases}$$



Here

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r.$$

$\ldots dxdy$

$\iint \ldots \underset{=}{r}\, dr d\theta$

Note that $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = |r| = r$, since $r \geq 0$.

To compute $\frac{\partial(r,\theta)}{\partial(x,y)}$, we write

$$r^2 = x^2 + y^2 ,$$

$$\theta = \tan^{-1}\frac{y}{x} ; \quad \text{then}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \frac{x^2+y^2}{r^3} = \frac{1}{r}.$$

i.e. $$\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{1}{\frac{\partial(r,\theta)}{\partial(x,y)}}.$$

$r\frac{\partial r}{\partial x} = 2x$

$\frac{\partial r}{\partial x} = \frac{x}{r}$

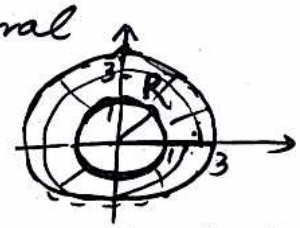
$\ldots = \frac{1}{1+(\frac{y}{x})^2} \cdot \frac{-y}{x^2}$

$= \frac{-y}{x^2+y^2}$

$= \frac{-y}{r^2}$

Thus, if we want to evaluate, by changing to polar coor., the following integral

$$\iint_R e^{x^2+y^2}\, dx\, dy ,$$



$= r\cos\theta$

$\cdot \sin\theta$

x^2+y^2

where R is the region between 2 circles $x^2+y^2=1$ and $x^2+y^2=9$, we can use (1) above, and obtain from (A), Pg. 113.

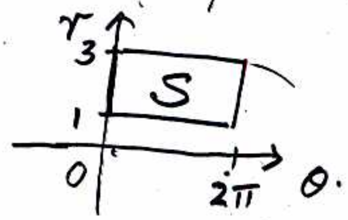
$$\iint_R e^{x^2+y^2}\,dx\,dy = \iint_S e^{r^2}\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta.$$

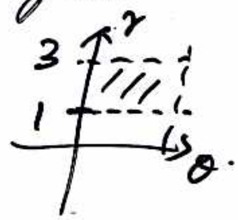
$$= \iint_S \underline{e^{r^2}} \cdot \underline{r\,dr\,d\theta}.$$

Here S is the corresponding region

$r^2 = x^2+y^2$

$\theta = \tan^{-1}\frac{y}{x}$





Hence

$$\iint_S e^{r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\int_1^3 \underline{e^{r^2}\,r\,dr}\,d\theta$$

$u = r^2$

$$= \int_0^{2\pi}\left[\frac{1}{2}e^{r^2}\right]_1^3 d\theta$$

$$= \int_0^{2\pi}\frac{1}{2}[e^9 - e^1]\,d\theta.$$

$$= \frac{1}{2}[e^9 - e]\,\theta\Big]_0^{2\pi}$$

$$= \frac{1}{2}(e^9 - e)2\pi$$

$$= (e^9 - e)\pi.$$

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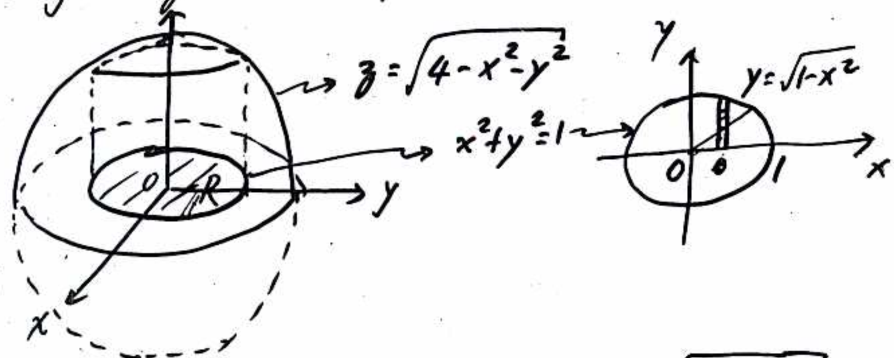
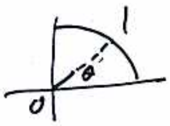
Ex. 2. Find the vol. of the solid cut out of the sphere $x^2+y^2+z^2=4$ by the cylinder $x^2+y^2=1$.



Consider the hemisphere $z=\sqrt{4-x^2-y^2}$ and the base $x^2+y^2=1$.

Then

$$\text{Vol.} = 2\iint_R \sqrt{4-x^2-y^2}\,dx\,dy$$

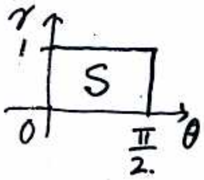


$$= 2\cdot 4\int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{4-x^2-y^2}\,dy\,dx$$

Let

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases}$$

$$= 8\int_0^{\pi/2}\int_0^1 \sqrt{4-r^2}\;r\,dr\,d\theta$$



$$= 8\int_0^{\pi/2} -\frac{1}{3}(4-r^2)^{3/2}\Big|_0^1\,d\theta.$$

Let

$u = 4-r^2$

$du = -2r\,dr$

$\frac{du}{-2} = r\,dr$

$\int\sqrt{4-r^2}\,r\,dr = -\frac{1}{2}\int\sqrt{u}\,du$

$= \frac{-u^{3/2}}{3}$

$= -\frac{1}{3}(4-r^2)^{3/2}$

$$= \frac{-8}{3}\int_0^{\pi/2} 3^{3/2}-4^{3/2}\,d\theta$$

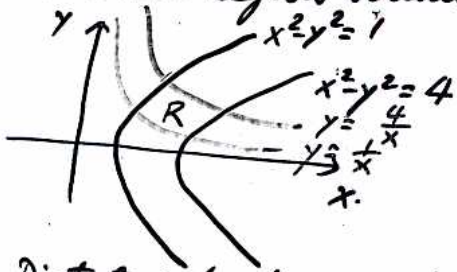
$$= -\frac{8}{3}\left(3^{3/2}-8\right)\cdot\frac{\pi}{2}.$$

$$= \frac{4\pi}{3}\left(8-3^{3/2}\right) \text{ cubic units.}$$

Ex. 3. Evaluate

$$\iint_R (x^2+y^2)\,dx\,dy.$$

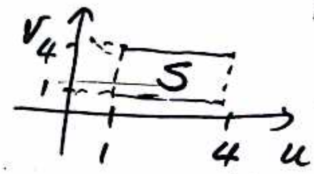
where R is the region bounded by the curves.



Soln: Dictated by the region R, we make the change of variables

$$u = x^2-y^2.$$
$$v = xy$$

Then the boundaries of R become $u=1$, $u=4$, $v=1$, $v=4$.



Also

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2+y^2)$$

So that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2(x^2+y^2)}.$$

Then

$$\iint_R (x^2+y^2)\,dx\,dy = \iint_S x^2+y^2 \cdot \frac{1}{2(x^2+y^2)}\,du\,dv$$
$$= \int_1^4\int_1^4 \frac{1}{2}\,du\,dv$$
$$= \frac{1}{2}\int_1^4 u\Big|_1^4\,dv = \frac{1}{2}\int_1^4 3\,dv$$
$$= \frac{1}{2}\cdot 3\cdot 3 = \frac{9}{2}.$$