Triple Integration

We can regard the triple integral

$$\iiint_V f(x,y,z)\,dx\,dy\,dz$$

as an extension of the double integral over a region S. One such volume V, in xyz-space, can be contained in the box bounded by 6 planes. (fig. 1)

$$x=a,\ x=b;\ y=c,\ y=d;\ z=e,\ z=f.$$

Another volume V can have a lower surface given by

$$z=f_1(x,y)$$

and an upper surface given by

$$z=f_2(x,y),$$

and laterally bounded by a cylinder C with base R on the xy-plane, i.e. the base R is the orthogonal projection of the solid into the xy-plane (see fig. 2).

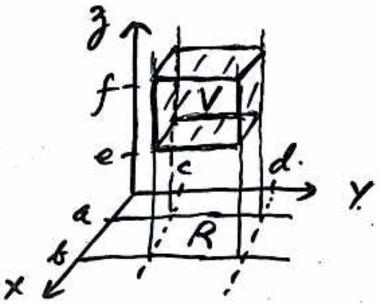


fig. 1

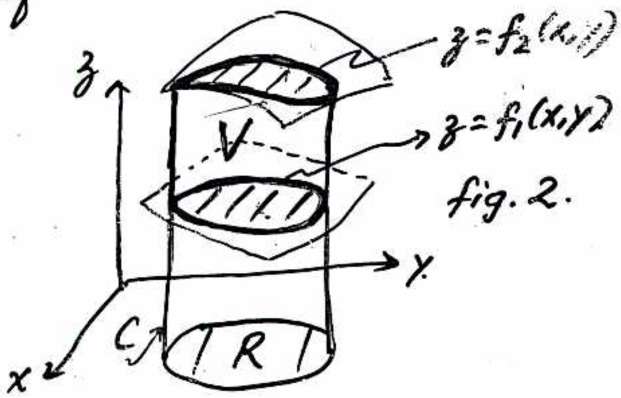


fig. 2.

In this case, we evaluate

$$\iiint_V f(x,y,z)\,dz\,dy\,dx$$

$$= \iint_R \int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dz \cdot dy\,dx.$$

Ex. 1 Evaluate the triple integral

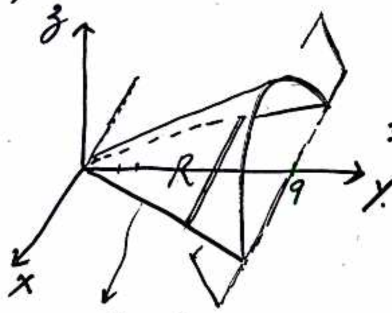
dV $\qquad \iiint\limits_V z\,dz\,dx\,dy$, where V is the volume above the xy plane bounded by the cone

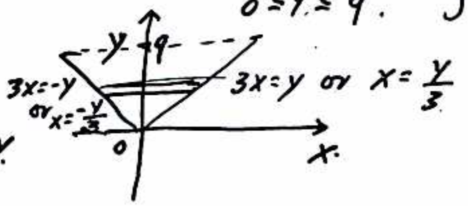
$$9x^2 + z^2 = y^2$$

and the plane $y = 9$.

Soln First, we sketch the volume V: $\left.\begin{array}{l} 0 \le z \le \sqrt{y^2 - 9x^2} \\ -\frac{y}{3} \le x \le \frac{y}{3} \\ 0 \le y \le 9. \end{array}\right\}$

At $z = 0$
$9x^2 = y^2$





$9x^2 = y^2$ or $3x = y$

$$\therefore \iiint\limits_V z\,dz\,dx\,dy = \int_0^9 \int_{-\frac{y}{3}}^{\frac{y}{3}} \int_0^{\sqrt{y^2-9x^2}} z\,dz\,dx\,dy$$

$$= \int_0^9 \int_{-\frac{y}{3}}^{\frac{y}{3}} \frac{1}{2} z^2 \Big|_0^{\sqrt{y^2-9x^2}} dx\,dy$$

$$= \int_0^9 \int_{-\frac{y}{3}}^{\frac{y}{3}} \frac{1}{2}(y^2 - 9x^2)\,dx\,dy$$

$$= \int_0^9 \frac{1}{2}\left(y^2 x - 3x^3\right)\Big|_{-\frac{y}{3}}^{\frac{y}{3}} dy$$

$$= \int_0^9 \frac{2}{9} y^3\,dy.$$

$$= \frac{2}{9 \cdot 4} y^4 \Big|_0^9 = \frac{729}{2}.$$

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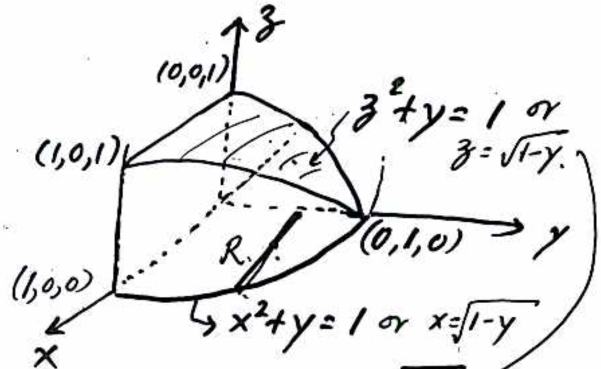
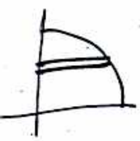
Ex. 2. Evaluate $\iiint_V y^2 \, dV$, where V is the volume (or region) bounded by the cylinders $x^2+y=1$ and $z^2+y=1$ and the plane $y=0$



The vol. in the first octant. Required volume is 4 x this volume in 1st octant.



$$\iiint_V y^2 \, dV = 4\int_0^1 \int_0^{\sqrt{1-y}} \int_0^{\sqrt{1-y}} y^2 \, dz \, dx \, dy.$$

$$= 4\int_0^1 \int_0^{\sqrt{1-y}} y^2 z \Big|_0^{\sqrt{1-y}} dx \, dy$$

$$= 4\int_0^1 \int_0^{\sqrt{1-y}} y^2 \sqrt{1-y} \, \underline{dx} \, dy.$$

Let
$u = 1-y$
$du = -dy$

$$= 4\int_0^1 y^2\sqrt{1-y}\,\sqrt{1-y} \, dy$$

$$= 4\int_0^1 y^2(1-y)\, dy$$

$$= 4\int_0^1 y^2 - y^3 \, dy$$

$$= 4\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{1}{3}.$$

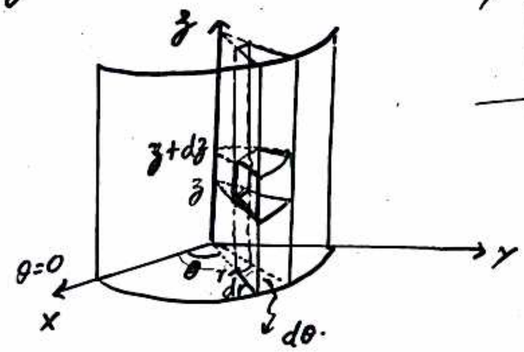
Triple Integral in Cylindrical Coordinates (r, θ, z).

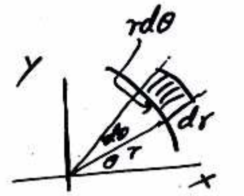
The change of variables from (x, y, z) to cylindrical coordinates (r, θ, z) is given by

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z.$$

Geometrically,

$r\,dr\,d\theta$





The Jacobian of the transformation is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \cdot 1$$

$$= r$$

Hence, if we wish to evaluate a triple integral by changing to cylindrical coordinates, we get

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V_1} f(r\cos\theta, r\sin\theta, z)\, \underline{r\,dr\,d\theta\,dz}$$

where V is now mapped to the volume (or region) V_1 in the new r, θ, z coordinates.

Triple Integral in Spherical Coordinates (ρ, θ, ϕ)

The relationship between (x, y, z) coor. and spherical coor. (ρ, θ, ϕ) is given by

$$x = \rho \cos\theta \cdot \sin\phi \qquad = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\theta \cdot \sin\phi \qquad = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

Just as above, we can compute its Jacobian

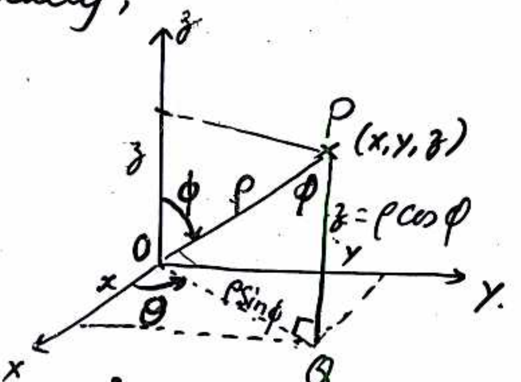
$$\left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right| = \underline{\rho^2 \sin\phi} \qquad (\text{do it!})$$

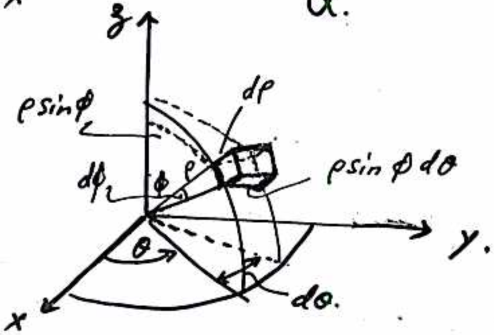
So that we now have

$$\iiint_V f(x,y,z)\, dV = \iiint_{V_2} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi) \cdot \underline{\rho^2 \sin\phi \cdot d\rho\, d\theta\, d\phi}.$$

where V is now mapped to the volume (or region) V_2 in ρ, θ, ϕ coordinates.

Geometrically,





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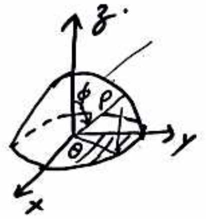
Ex. 1 Evaluate $\iiint xyz\,dV$, where V is the solid in the first octant bounded by the sphere $x^2+y^2+z^2=16$ by (a) using spherical coor., (b) using cylindrical coor.

Soln (a) In spherical coor.

$$x=\rho\sin\phi\cos\theta$$
$$y=\rho\sin\phi\sin\theta$$
$$z=\rho\cos\phi,$$



$$dV=\underline{\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

and the sphere (in the 1st octant) becomes

$$\rho^2=16$$

$$V_1=\begin{cases}0\le\rho\le 4\\ 0\le\phi\le\frac{\pi}{2}\\ 0\le\theta\le\frac{\pi}{2}.\end{cases}$$

Thus we have

$$\iiint_V xyz\,dV=\iiint_{V_1}\rho\sin\phi\cos\theta\cdot\rho\sin\phi\sin\theta\cdot\rho\cos\theta\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^4\rho^5\underline{\sin^3\phi\cos\phi}\cos\theta\sin\theta\,d\rho\,d\phi\,d\theta$$

Let $u=\sin\phi$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{6}\rho^6\Big|_0^4\underline{\sin^3\phi\cos\phi}\cos\theta\sin\theta\,d\phi\,d\theta$$

$$=\frac{1}{6}\cdot4^6\int_0^{\pi/2}\underline{\frac{1}{4}\sin^4\phi\Big|_0^{\pi/2}}\cos\theta\sin\theta\,d\theta$$

$$=\frac{1}{3}\cdot2^{11}\cdot\frac{1}{4}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta$$

$$=\frac{1}{3}\cdot2^9\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}$$

$$=\frac{1}{3}\cdot2^9\cdot\frac{1}{2}=\frac{1}{3}\cdot2^8=\frac{256}{3}.$$

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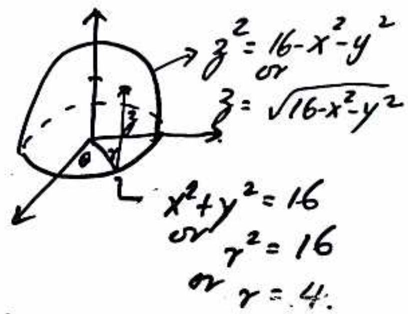
Solu (b) In cylindrical coor.

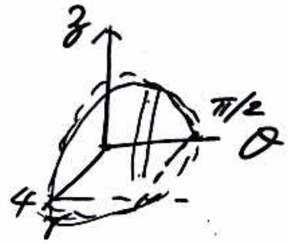
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z.$$



$dV = \underline{r\,dr\,d\theta\,dz}$

and the sphere (in 1st octant) becomes

$$V_2 = \begin{cases} 0 \le z \le \sqrt{16-x^2-y^2} = \sqrt{16-r^2} \\ 0 \le r \le 4 \\ 0 \le \theta \le \frac{\pi}{2}. \end{cases}$$



Thus

$$\iiint_V xyz\,dV = \iiint_{V_2} r\cos\theta \cdot r\sin\theta \cdot z \cdot \; r\,dr\,d\theta\,dz.$$

$$= \iiint_{V_2} r^3\cos\theta\sin\theta \cdot z \cdot dr\,d\theta\,dz$$

$$= \int_0^{\pi/2}\int_0^4\int_0^{\sqrt{16-x^2-y^2}=\sqrt{16-r^2}} r^3\cos\theta\sin\theta\, z\; dz\,dr\,d\theta.$$

$$= \int_0^{\pi/2}\int_0^4 r^3\cos\theta\sin\theta\; \frac{1}{2}z^2\Big|_0^{\sqrt{16-r^2}}\; dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\int_0^4 \cos\theta\sin\theta\; r^3(16-r^2)\; dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos\theta\sin\theta \left[4r^4 - \frac{r^6}{6}\right]_0^4 d\theta$$

$\frac{4^6}{6} = \frac{4 \cdot 4^5}{6} = \frac{2}{3}4^5$

$$= \frac{1}{2}\left[4^5 - \frac{4^6}{6}\right]\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta$$

$$= \frac{1}{2}\left[\frac{4^5}{3}\right]\;\frac{1}{2}\sin^2\theta\Big|_0^{\pi/2}$$

$$= \frac{1}{4}\left[\frac{4^5}{3}\right] = \frac{4^4}{3} = \frac{256}{3}.$$

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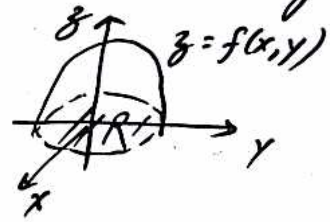
Area of A Surface

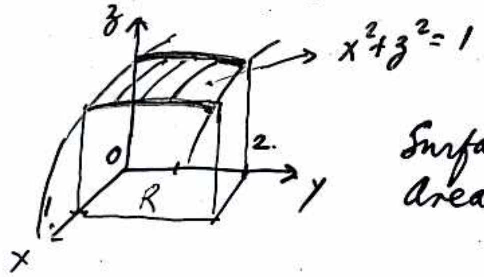
Suppose $z = f(x,y)$ is defined over a closed region R in the xy-plane.



We wish to find the surface area. It can be shown that the

$$\longrightarrow \quad \text{Surface Area} = \iint\limits_R \sqrt{1 + z_x^2 + z_y^2}\; dx\,dy.$$

Ex. 1. Find the area of the surface of the cylinder $x^2 + z^2 = 1$ that is cut by the planes $x = 0$, $x = 1$, $y = 0$ and $y = 2$.



$$x^2 + z^2 = 1$$

$$2x + 2z\frac{\partial z}{\partial x} = 0$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{x}{z}$$

$$\therefore z_x^2 = \frac{x^2}{z^2}$$

$$\begin{aligned}
\text{Surface area} &= \int_0^2\int_0^1 \sqrt{1 + z_x^2 + z_y^2}\; dx\,dy \\
&= \int_0^2\int_0^1 \sqrt{1 + \frac{x^2}{1-x^2} + 0}\; dx\,dy \\
&= \int_0^2\int_0^1 \sqrt{\frac{1}{1-x^2}}\; dx\,dy \\
&= \int_0^2\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx\,dy \\
&= \int_0^2 \sin^{-1}x \Big|_0^1\, dy \\
&= \int_0^2 \sin^{-1}1\; dy \\
&= \sin^{-1}1 \cdot y\Big|_0^2 \\
&= 2\sin^{-1}1 = 2\cdot\left(\frac{\pi}{2}\right) = \pi \text{ sq. units.}
\end{aligned}$$

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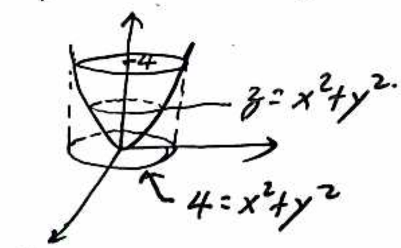
<u>Ex. 2.</u> Find the area of the paraboloid $z = x^2+y^2$ below the plane $z=4$.

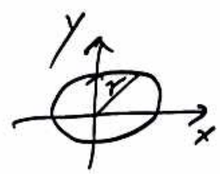


The closed region in the xy plane is bounded by $x^2+y^2=4$. $\quad z_x = 2x,\ z_y = 2y$.

$$\therefore \text{Surface Area} = \iint_R \sqrt{1+(2x)^2+(2y)^2}\,dx\,dy$$

$$= \iint_R \sqrt{1+4(x^2+y^2)}\,dx\,dy.$$

Changing to polar coor.



$x = r\cos\theta$

$y = r\sin\theta$.

$$\therefore \text{Surface Area} = \int_0^{2\pi}\int_0^{2} \sqrt{1+4r^2}\; r\,dr\,d\theta$$

Let $u = 1+r^2$.

$$= \int_0^{2\pi} \frac{1}{12}(1+4r^2)^{3/2}\Big|_0^2\, d\theta.$$

$$= \frac{1}{12}(17^{3/2}-1)\int_0^{2\pi} d\theta$$

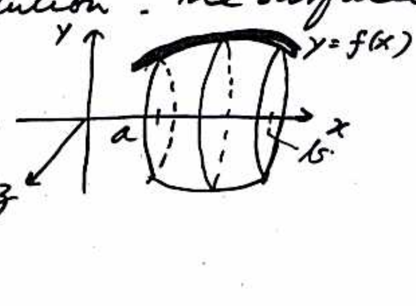
$$= \frac{1}{12}(17^{3/2}-1)\cdot 2\pi.$$

$$= \frac{\pi}{6}(17\sqrt{17}-1).$$

<u>Area of a Surface of Revolution</u>

If a curve <u>$y=f(x)>0$</u> is rotated about the x-axis we obtain a surface of revolution. The surface's eq. is

→ $\quad y^2+z^2 = [f(x)]^2$

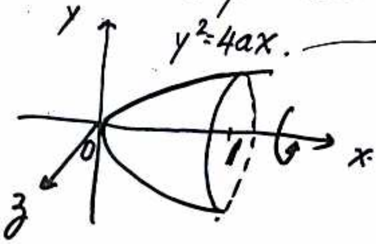


g.125 The area of this surface of revolution is given by

$$\longrightarrow \quad \text{Surface Area} = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx,$$

where $f' = \dfrac{df}{dx} = \dfrac{dy}{dx}$

Ex. 1. Find the area of the paraboloid generated by revolving the parabola $y^2 = 4ax$, with $0 \le x \le 1$, about x-axis.



$$y^2 = 4ax \longrightarrow y = \sqrt{4ax}$$

$$\therefore y' = 2\sqrt{a}\,\frac{1}{2\sqrt{x}} = \sqrt{\frac{a}{x}}$$

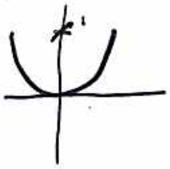
$$\text{Surface Area} = 2\pi \int_0^1 \sqrt{4ax}\sqrt{1+\frac{a}{x}}\,dx$$

$$= 2\pi\, 2\sqrt{a} \int_0^1 \sqrt{x+a}\,dx$$

$$= 4\pi\sqrt{a}\cdot\frac{2}{3}(x+a)^{3/2}\Big|_0^1$$

$$= \frac{8\pi\sqrt{a}}{3}\left[(1+a)^{3/2} - a^{3/2}\right] \text{ sq. units}$$

Ex. 2 Find the area of the surface of revolution obtained by revolving about the x-axis the catenary $y = a\cosh\dfrac{x}{a}$, with $0 \le x \le a$.



$$\text{Here, } y' = \frac{a}{a}\sinh\frac{x}{a} = \sinh\frac{x}{a}.$$

$$\sqrt{1+y'^2} = \sqrt{1+\sinh^2\frac{x}{a}} = \cosh\frac{x}{a}.$$

$$\therefore \text{Surface Area} = 2\pi \int_0^a a\cosh\frac{x}{a}\cdot\cosh\frac{x}{a}\,dx$$

$$= 2\pi a \int_0^a \cosh^2\frac{x}{a}\,dx$$

$$= 2\pi a\cdot\frac{1}{2}\int_0^a \left(1+\cosh\frac{2x}{a}\right)dx$$

$$= \pi a\left[x + \frac{a}{2}\sinh\frac{2x}{a}\right]_0^a$$

$$= \pi a\left(a + \frac{a}{2}\sinh 2\right)$$

END OF LECTURE. GOOD LUCK.

$$= \pi a^2\left(1 + \frac{1}{2}\sinh 2\right) \text{ square units.}$$