# Green's Theorem

Let $F(x,y) = (f(x,y), g(x,y))$ be a vector field on an open set $U$ in the xy-plane.

Let $C$ be a curve in $U$.

Then the curve integral is

$$\int_C F = \int_C f(x,y)\,dx + g(x,y)\,dy.$$

If the curve $C$ is a closed path, in a counterclockwise direction, and enclosing a region $A$, then we have the result —

## Green's Th.


$$\oint_C f(x,y)\,dx + g(x,y)\,dy$$

$$= \iint_A \left( -\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} \right) dA.$$

dxdy or dydx

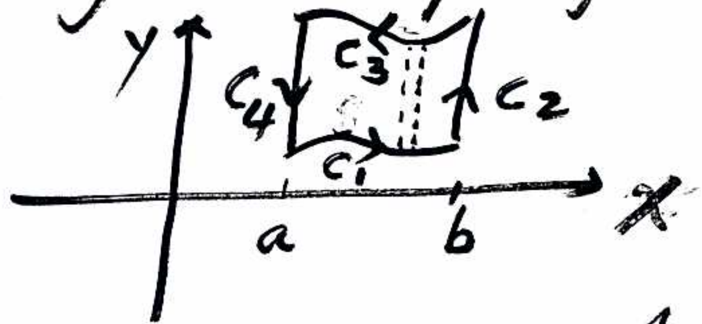
The counterclockwise closed path $C$ can take various shapes, such as


and the region $A$ is to <u>the left</u> of path $C$.

If closed path consists of 2 circles $C_1$ & $C_2$, then


the inner closed path $C_2$ has to be <u>oriented</u> clockwise, so that the enclosed region is to the left of $C_2$ (and also to the left of $C_1$).

<u>Proof</u>: We only show the proof for this case:




$C = \{c_1, c_2, c_3, c_4\}$, where

$c_1$: $y = f_1(x)$, $a \le x \le b$

$c_2$: $x = b$,

$c_3$: $y = f_2(x)$, $a \le x \le b$

$c_4$: $x = a$.

Let us prove that

$$\int_C f\,dx = \iint_A -\frac{\partial f}{\partial y}\,dy\,dx.$$

We see that

$$\iint_A + \frac{\partial f}{\partial y}\,dy\,dx = \int_a^b f(x,y)\Big]_{f_1(x)}^{f_2(x)}\,dx$$

$$= \int_a^b f(x, f_2(x)) - f(x, f_1(x))\,dx$$

$$= \int_{c_3^-} f(x,y)\,dx - \int_{c_1} f(x,y)\,dx$$

$$= -\int_{C_3} f(x,y)\,dx - \int_{C_1} f(x,y)\,dx$$

$$= -\oint_C f(x,y)\,dx,$$

Since $\int_{C_2} f(x,y)\,dx = 0 = \int_{C_4} f(x,y)\,dx$ $\left(\because \begin{array}{l} x = a \text{ or } b \\ dx = 0 \end{array}\right)$

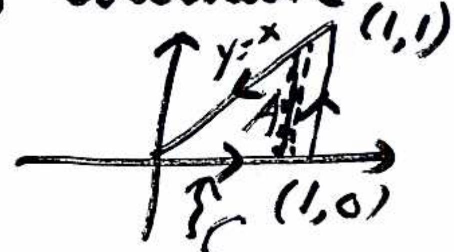
Similarly, we can show:

$$\oint_C g(x,y)\,dy = \iint_A \frac{\partial g}{\partial x}\,dx\,dy.$$

Hence Green's Th. is proved.

Ex. Use Green's Th. to evaluate

$$\oint_C y^2\,dx + x\,dy$$




$$= \iint_A (1-2y)\,dy\,dx$$

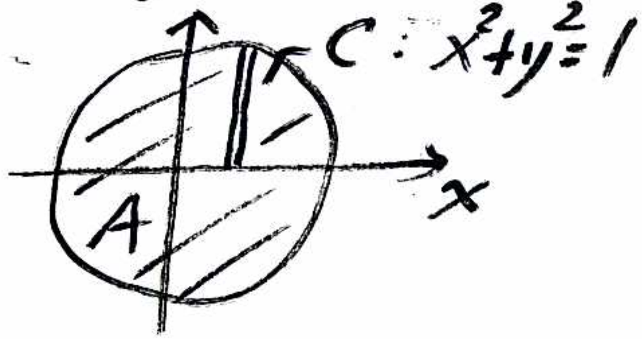
$$= \int_0^1 \int_{y=0}^{y=x} (1-2y)\,dy\;dx$$

$$= \int_0^1 y - y^2\Big]_0^x\,dx = \int x - x^2\,dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3}\Big]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Ex. 2. Use Green's Th. to find (G5)

$$\oint_C y^2dx + xdy$$




$$= \iint_A 1-2y \, dydx$$

We use polar coor: $x = r\cos\theta$, $y = r\sin\theta$, $0 \le \theta \le 2\pi$


$$= \iint_R (1-2r\sin\theta) r dr d\theta$$

$$= \int_0^{2\pi}\int_0^1 r - 2r^2\sin\theta \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2} - \frac{2r^3}{3}\sin\theta \Big]_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} - \frac{1}{3}\sin\theta \, d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{3}\cos\theta \Big]_0^{2\pi}$$

$$= \frac{2\pi}{2} + \frac{1}{3}\cos 2\pi - \frac{1}{3}\cos 0$$

$$= \pi.$$