05-05-25

MATH 331 (L20) Name:

Mid Term ID: ________

Total marks : 25

Marks

[3] 1(a) Solve $(2y^2+6xy)dx+(3x^2+4xy+\sin 2y)dy=0$

$$\frac{\partial M}{\partial y}=4y+6x=\frac{\partial N}{\partial x} \quad \therefore \text{Exact.}$$

Integrating

$$\int 2y^2+6xy\,\partial x+\int 3x^2+4xy+\sin 2y\,\partial y=c$$

$$2y^2x+3x^2y+\cancel{3x^2y}+\cancel{2xy^2}-\frac{1}{2}\cos 2y=c$$

[4] (b) Solve $\frac{dy}{dx}=\frac{2y}{x}+x^3e^x.$

$$\frac{dy}{dx}-\frac{2y}{x}=x^3e^x \qquad \text{linear eq.}$$

$$IF=e^{-\int\frac{2}{x}dx}=e^{-2\ln x}=e^{\ln x^{-2}}$$

$$=x^{-2}.$$

The general solution is

$$x^{-2}y=\int x^{-2}\cdot x^3e^x dx=\int xe^x dx+c$$

$$=xe^x-\int e^x dx+c \qquad \text{(by parts)}$$

$$=xe^x-e^x+c$$

$$\therefore y=x^2(xe^x-e^x+c)$$

[6] 2. Solve $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ homogeneous

Let $y = ux$

$\left({}' = \frac{d}{dx} \right)$ Then $y' = u'x + u = \dfrac{1+u}{1-u}$

$$u'x = \frac{1+u}{1-u} - u = \frac{1+u-u(1-u)}{1-u}$$

$$= \frac{1+u^2}{1-u}$$

$$\therefore \frac{1-u}{1+u^2}\,du = \frac{dx}{x}$$

$$\int \frac{1}{1+u^2} - \frac{u}{1+u^2}\,du = \int \frac{dx}{x} + c$$

$$\tan^{-1} u - \frac{1}{2}\ln(1+u^2) = \ln|x| + c$$

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln|x| + c$$

$\downarrow$ can be simplified

$$-\frac{1}{2}\ln\frac{x^2+y^2}{x^2}$$

$$= -\frac{1}{2}\left[\ln|x^2+y^2| - 2\ln|x|\right]$$

$$\therefore \tan^{-1}\frac{y}{x} - \frac{1}{2}\ln(x^2+y^2) = c$$ (cancelling out $\ln|x|$).

[6] 3. Solve $x^2\frac{dy}{dx} + xy = 3y^3$, $y(1) = 3$ Bernoulli's Eq.

$\div x^2y^3$

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{xy^2} = \frac{3}{x^2}$$

Let $u = \frac{1}{y^2}$

Then $\frac{du}{dx} = \frac{-2}{y^3}\frac{dy}{dx}$

$$\therefore \frac{1}{-2}\frac{du}{dx} + \frac{u}{x} = \frac{3}{x^2}$$

$$\frac{du}{dx} - \frac{2u}{x} = \frac{-6}{x^2} \qquad \text{linear.}$$

$$I.F. = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = x^{-2}.$$

The general solution is

$$x^{-2}\cdot u = \int \frac{-6}{x^2}\cdot x^{-2}dx = \int -6x^{-4}dx$$

$$= \frac{-6x^{-3}}{-3} + c = 2x^{-3} + c$$

$$u = x^2(2x^{-3} + c) = \frac{2}{x} + cx^2$$

$$\text{or } \frac{1}{y^2} = \frac{2}{x} + cx^2 \quad \left(\text{or } \frac{2 + cx^3}{x}\right)$$

At $x=1$, $9 = \frac{1}{2+c}$ ie $y^2 = \frac{x}{2+cx^3}.$

$c = \frac{1}{9} - 2 = \frac{-17}{9}$ $y^2 = \frac{x}{\ldots} = \frac{9x}{18-17\ldots}$

[6] 4. Consider the family of curves $cy^2 = x^2 - 1$, where c is an arbitrary constant. Find its orthogonal trajectories

$$cy^2 = x^2 - 1$$

$$c \cdot 2y\frac{dy}{dx} = 2x. \quad \therefore \frac{dy}{dx} = \frac{2x}{2cy} = \frac{x}{\frac{x^2-1}{y}} = \frac{xy}{x^2-1}$$

$\therefore$ O.T. are given by

$$\frac{dy}{dx} = \frac{-(x^2-1)}{xy} = \frac{1-x^2}{xy} \qquad \text{Separable}$$

$$\int y\,dy = \int \frac{1-x^2}{x}\,dx = \int \frac{1}{x} - x\,dx$$

$$\frac{y^2}{2} = \ln|x| - \frac{x^2}{2} + C$$