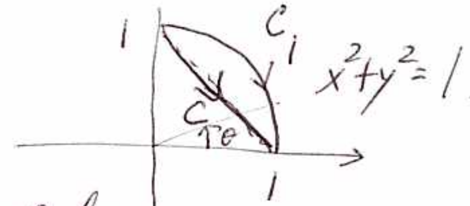
Spring '05 MATH 331 (L20) p1

Assignment 5 (Solution)

1. (a) #15 p227 Easier way to solve is to use a change of variables from x, y to r, θ i.e.

$$x = r\cos\theta$$
$$y = r\sin\theta.$$



Then

$$dx = -r\sin\theta\, d\theta + \cos\theta\, dr$$
$$dy = r\cos\theta\, d\theta + \sin\theta\, dr.$$

$$\therefore \frac{-y}{x^2+y^2}dx = \frac{-r\sin\theta}{r^2}(-r\sin\theta\, d\theta + \cos\theta\, dr)$$

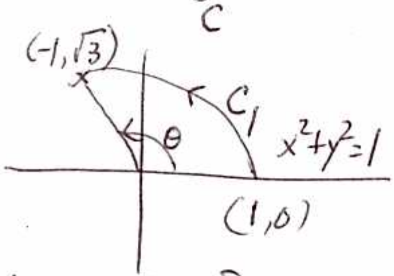
$$\frac{x}{x^2+y^2}dy = \frac{r\cos\theta}{r^2}(r\cos\theta\, d\theta + \sin\theta\, dr)$$

Adding:

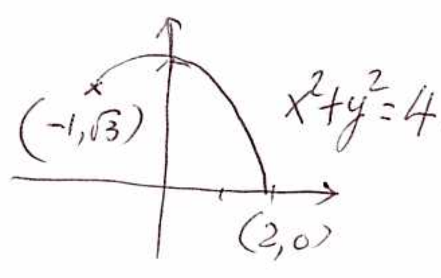
$$\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \frac{r^2(\sin^2\theta + \cos^2\theta)\, d\theta}{r^2}$$
$$= d\theta.$$

(a) $\therefore \displaystyle\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \int_{C_1} d\theta = \int_{\pi/2}^{0} d\theta = -\frac{\pi}{2}.$

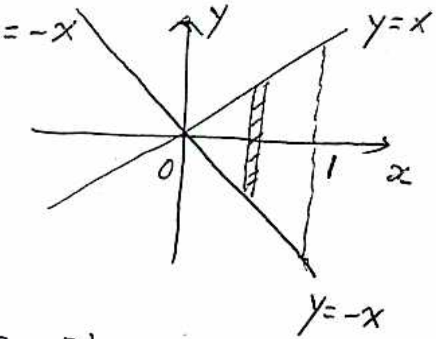
(b) $\therefore \displaystyle\int_C$ " $\displaystyle = \int_{C_1} d\theta = \int_0^{2\pi/3} d\theta = \theta\Big]_0^{2\pi/3} = \frac{2\pi}{3}.$



(c) Try it.



2.
8(b) p251



$x^2-y^2 \geq 0$

$(x+y)(x-y) \geq 0$ (2 lines $y+x=0$, $x-y=0$)

Also $0 \leq x \leq 1$

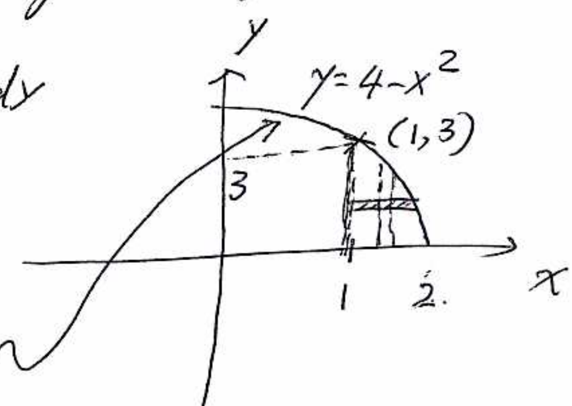
$$\int_0^1 \int_{-x}^{x} (x^2-y^2)\,dy\,dx$$

$$= \int_0^1 \left. x^2y - \frac{y^3}{3} \right]_{-x}^{x} dx = \int_0^1 x^3 - \frac{x^3}{3} - \left(-x^3 + \frac{x^3}{3}\right) dx$$

$$= \int_0^1 2x^3 - \frac{2}{3}x^3\,dx = \int_0^1 \frac{4}{3}x^3\,dx = \left. \frac{\cancel{4}x^4}{\cancel{4}\cdot 3} \right]_0^1 = \frac{1}{3}.$$

3. First, sketch the region of integration, from

$$\int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} (x+y)\,dx\,dy$$



We see that $\longrightarrow$ $x=\sqrt{4-y}$

$x^2 = 4-y$

$y = 4-x^2.$

At $x=1$, $y=3$. At $y=0$, $x=2$.

Changing order of integration (ie to integrate wrt. y first).

$$\int_1^2 \left(\int_{y=0}^{4-x^2} (x+y)\,dy \right) dx$$

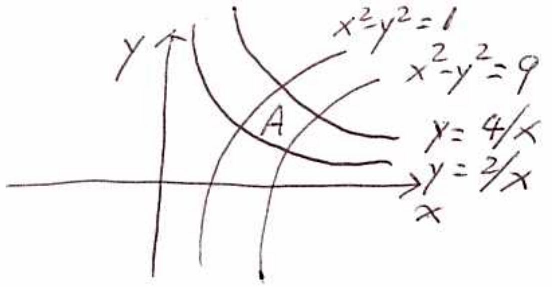
$$= \int_1^2 \left. xy + \frac{y^2}{2} \right]_0^{4-x^2} dx = \int_1^2 x(4-x^2) + \frac{1}{2}(4-x^2)^2\,dx$$

$$= \int_1^2 4x - x^3 + \frac{1}{2}(16 - 8x^2 + x^4)\,dx$$

$$= \left. 2x^2 - \frac{x^4}{4} + \frac{1}{2}\left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right) \right]_1^2 =$$

$$= 8 - 4 + \frac{1}{2}\left(32 - \frac{64}{3} + \frac{32}{5}\right) - \left[2 - \frac{1}{4} + \frac{1}{2}\left(16 - \frac{8}{3} + \frac{1}{5}\right)\right] = 4\frac{1}{60}$$

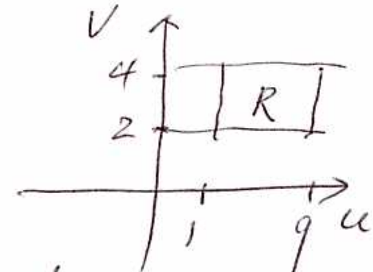
4.



Change variables from (x,y) to (u,v):

$$u = x^2-y^2$$
$$v = xy$$

Then we have the region R:



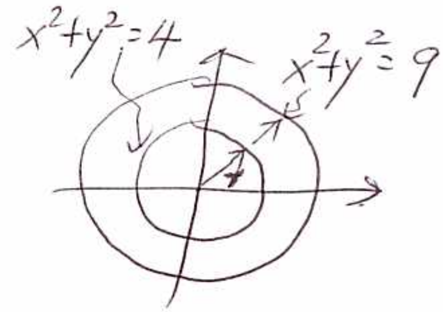
$$I = \iint_A (x^2+y^2)\,dx\,dy = \iint_R (x^2+y^2)\,|J|\,du\,dv,$$

where
$$J = \underset{\substack{\uparrow \\ \text{can't be obtained}}}{\frac{\partial(x,y)}{\partial(u,v)}} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}}$$

$$= \frac{1}{\begin{vmatrix} 2x & y \\ -2y & x \end{vmatrix}} = \frac{1}{2(x^2+y^2)}.$$

$$\therefore I = \iint_R (x^2+y^2)\cdot\frac{1}{2(x^2+y^2)}\,du\,dv$$
$$= \int_2^4\int_1^9 \frac{1}{2}\,du\,dv = \frac{1}{2}\cdot(9-1)\cdot(4-2) = 8$$

5. $\iint_A \sqrt{x^2+y^2}\,dx\,dy$. Let



$$x = r\cos\theta$$
$$y = r\sin\theta.$$

$$= \int_0^{2\pi}\int_{r=2}^{3} r\cdot r\,dr\,d\theta \qquad J = \frac{\partial(x,y)}{\partial(r,\theta)} = r.$$

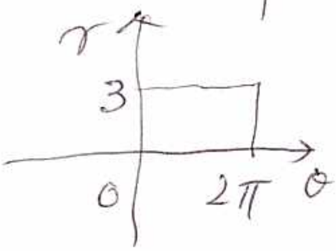
$$= \int_0^{2\pi} \frac{r^3}{3}\Big]_2^3\,d\theta = \left(9-\frac{8}{3}\right)\cdot 2\pi = \frac{38\pi}{3}.$$

6. Green's Th.

$$\oint_C f\,dx + g\,dy = \iint_A \left(-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x}\right) dx\,dy$$

(i) $C: x^2 + y^2 = 9$.

$$\oint y^2 dx + x\,dy = \iint_A (-2y+1)\,dx\,dy$$



Let $x = r\cos\theta$
$y = r\sin\theta$
$\therefore J = r$.

$$= \int_0^{2\pi}\int_0^3 (-2r\sin\theta + 1)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} -2\frac{r^3}{3}\sin\theta + \frac{r^2}{2}\Big]_0^3 d\theta$$

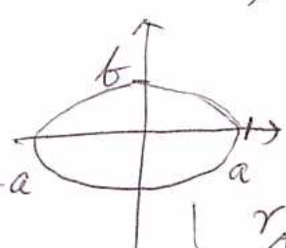
$$= \int_0^{2\pi} -18\sin\theta + \frac{9}{2}\,d\theta = 18\cos\theta + \frac{9}{2}\theta\Big]_0^{2\pi}$$

$\cos 0 = 1$
$\cos 2\pi = 1$

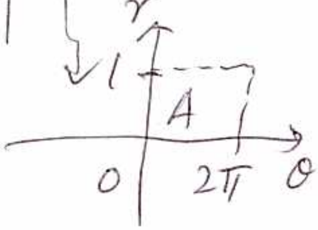
$$= 18\cos 2\pi + 9\pi - (18\cos 0)$$

$$= 9\pi.$$

(ii) $C: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Let. $x = a r\cos\theta$
$y = b r\sin\theta$. $\quad J = \frac{\partial(x,y)}{\partial(r,\theta)} = abr.$

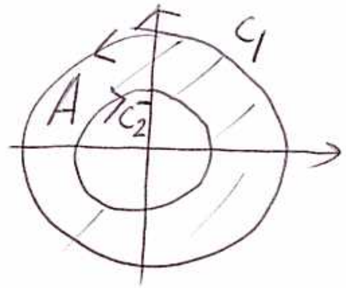


$$\therefore \oint_C y^2 dx + x\,dy = \iint_A (-2y+1)\,abr\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (-2br\sin\theta + 1)\,abr\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 ab(-2br^2\sin\theta + r)\,dr\,d\theta$$

$$= \int_0^{2\pi} ab\left(-2b\frac{r^3}{3}\sin\theta + \frac{r^2}{2}\right)\Big]_0^1 d\theta = ab\pi.$$

7. 

The region A is always to the left of curves c_1 + c_2^-, as shown. (imagine walking along c_1 + along c_2^-).

$$C = \{c_1, c_2^-\}.$$

By Green's Theorem,

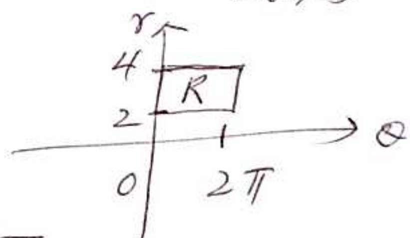
$c_1 = x^2+y^2=16$
$c_2 : x^2+y^2=4.$

$$\oint_C (x^3 - x^2 y)\,dx + xy^2\,dy$$

$$= \iint_A (x^2 + y^2)\,dx\,dy$$

Let $x = r\cos\theta$
$y = r\sin\theta$
then $J = \frac{\partial(x,y)}{\partial(r,\theta)} = r.$

$$= \int_0^{2\pi}\int_2^4 r^2 \cdot r\,dr\,d\theta$$



$$= \int_0^{2\pi} \frac{r^4}{4}\Big]_2^4 \,d\theta$$

$$= \int_0^{2\pi} 64-4\,d\theta = 120\pi.$$