Gradient Vector

For the fn. $f(x,y,z)$, the vector

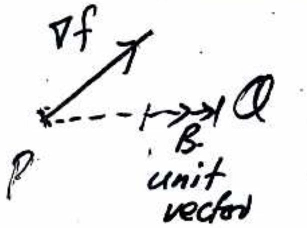
∇ del $\rightarrow$ $$\text{grad } f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

is called the gradient of f.

Thus, for $f = x^2+y^2+z^2-1$, then the grad of f is

$\nabla f|_P$ $$\text{grad } f = \nabla f = (2x, 2y, 2z).$$

Directional Derivative of f in the direction of $\vec{B}$.



For the fn $f(x,y,z)$, we have the grad. vector

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

If $\vec{B}$ is a unit vector, then we define the directional derivative of f in the dir^n of $\vec{B}$ to be the dot product of ∇f and $\vec{B}$, namely,

$\rightarrow$ $$D_B f(x,y,z) = \nabla f \cdot \vec{B}$$

(It measures the rate of change of f in the direction of $\vec{B}$).

Ex. 1 If $f = 3x^2 + xy - 2y^2 + z^2 - yz$, find the directional derivative of f at the point $P(1,-2,-1)$ in the direction of the vector $\vec{B}(2,-2,-1)$.

First, find the unit vector in the dir^n of $\vec{B}$, namely, $\dfrac{\vec{B}}{|\vec{B}|} = \dfrac{1}{\sqrt{2^2+2^2+1^2}}(2,-2,-1) = \frac{1}{3}(2,-2,-1$

Also, $\nabla f = (6x+y,\ x-4y-z,\ 2z-y)$

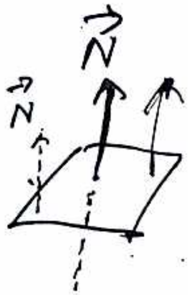
at P, $\nabla f = (4,\ 10,\ 0)$.

$\therefore \nabla f|_P \cdot \dfrac{\vec{B}}{|\vec{B}|} = (4,10,0)\cdot\frac{1}{3}(2,-2,-1) = \frac{1}{3}(8-20) = -$

Pg. 89. Note that (3)+(4) are of the form $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$.

Ex. 1. If a plane is given by

$$ax+by+cz=d$$



where a, b, c, d are constants; then comparing to the above eq., the normal to this plane is

$$\vec{N}=(a,\ b,\ c).$$

(any multiple of $\vec{N}$ can be a normal vector)

Ex. 2. Find the eq of the tangent plane to the surface (a) $xy+yz+zx=3$,

(b) $z=xy+x^2$,

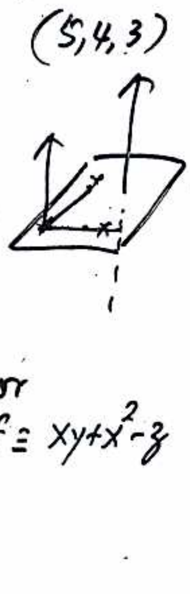
at the point $(1, 2, 3)$.

Soln (a) Let $f(x,y,z)\equiv xy+yz+zx-3=0$ be the surface. Then its normal at $\underline{(1,2,3)}$ is

$$\nabla f\Big|_{(1,2,3)}=(y+z,\ x+z,\ y+x)\Big|_{(1,2,3)}$$

$$=(5,\ 4,\ 3).$$

The reqd. tangent plane is



(3,) $\quad 5(x-1)+4(y-2)+3(z-3)=0$,

since it passes through $(1,2,3)$. Simplifying,

$$\underline{5x+4y+3z}=5+8+9=\underline{22}.$$

or $f\equiv xy+x^2-z$

(b) For the surface $z=xy+x^2$, the normal is

$$\vec{N}=(z_x,\ z_y,\ -1)=(y+2x,\ x,\ -1).$$

at $(1,2,3)$, $\vec{N}\Big|_{(1,2,3)}=(4,1,-1)$;

+ the tangent plane is

$$4(x-1)+1.(y-2)-(z-3)=0$$

or $\underline{4x+y-z}=4+2-3=\underline{3}$ ✓

Second-Derivative Test for Rel. Max/Min & Saddle

* **Theorem** Suppose $f(x,y)$ and its first and second partials are cts in some region, and suppose $P(x_0, y_0)$ is a critical point, i.e.

$$\begin{cases} f_x(x_0, y_0) = 0 \\ f_y(x_0, y_0) = 0 \end{cases}$$

Then

(i) $f(x,y)$ has a rel. min. value at $P(x_0, y_0)$, if

Discriminant $\quad D = f_{xx} f_{yy} - f_{xy}^2 \Big|_P > 0$ and $f_{xx} > 0$ (or $f_{yy} > 0$),

(ii) $f(x,y)$ has a rel. max value at $P(x_0, y_0)$, if

$$D = f_{xx} f_{yy} - f_{xy}^2 \Big|_P > 0 \text{ and } f_{xx} < 0 \text{ (or } f_{yy} < 0)$$

(iii) $f(x,y)$ has a saddle point at $P(x_0, y_0)$, if

$$D = f_{xx} f_{yy} - f_{xy}^2 \Big|_P < 0$$

(iv) further study is needed if

$$D = 0.$$

Ex. 1 Given $z = 2x^4 + y^2 - x^2 - 2y$ (see Ex. 1 on pg. 91).

We get

$$z_x = 8x^3 - 2x$$
$$z_y = 2y - 2.$$

The c.p. are $(0,1)$, $(\frac{1}{2}, 1)$ & $(-\frac{1}{2}, 1)$.

Also,

$$z_{xx} = 24x^2 - 2$$
$$z_{yy} = 2$$
$$z_{xy} = 0.$$

Then for each c.p., we have

Critical pt.	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - f_{xy}^2$	Conclusion
$(0,1,-1)$	-2	2	0	-4	f has a saddle pt.
$(\frac{1}{2},1,-1\frac{1}{8})$	4	2	0	8	f has a rel. minimum
$(-\frac{1}{2},1,-1\frac{1}{8})$	4	2	0	8	f has a rel. min. pt

Ex. 2. (a) For the fn $z = x^2 - 2y^2$, (see Ex. 2 on pg. 91)

$$z_x = 2x$$
$$z_y = -4y$$

We have 1 c.p. $(0,0,0)$.

Also
$$z_{xx} = 2$$
$$z_{yy} = -4$$
$$z_{xy} = 0$$

so that $f_{xx}f_{yy} - f_{xy}^2\Big|_{(0,0,0)} = -8$

By part (iii) of Theorem, the c.p. $(0,0,0)$ is a saddle pt.

(b) For the fn. $z = y(x - x^3)$, we have

$$z_x = y(1 - 3x^2)$$
$$z_y = x - x^3$$

and the c.p. are $(0,0)$, $(1,0)$ & $(-1,0)$

Then
$$z_{xx} = -6xy$$
$$z_{yy} = 0$$
$$z_{xy} = 1 - 3x^2.$$

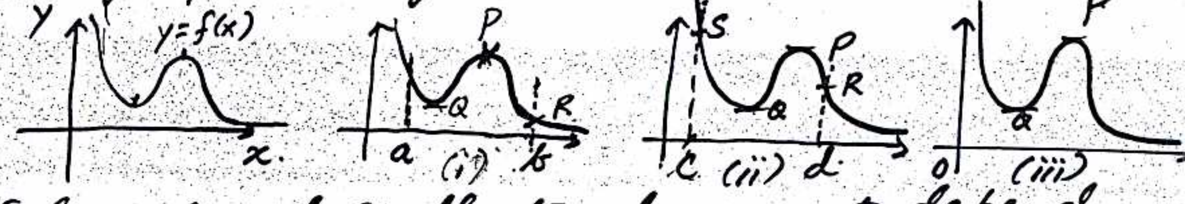
Thus.

Critical pt.	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	Conclusion.
$(0,0,0)$	0	0	1	-1	all are saddle pts.
$(1,0,0)$	0	0	-2	-4	
$(-1,0,0)$	0	0	-2	-4	

Absolute Max + Absolute Min. [Largest + Smallest Values]

If a fn $f(x,y)$ is defined in a domain D in the xy-plane, we can discuss not only its relative extremum values (around some critical points), but its largest and its smallest values over the entire domain D.

As examples, let us look at fns of one variable whose graphs are given:



Its largest and smallest values must depend on the interval of x.

In (i), the interval is $[a, b]$. (closed interval)
From the graph, its largest (or absolute max.) value is at P, which also gives the rel. max. value.
But its rel. min. value at Q is not its smallest value — at R.

In (ii), in the different closed interval $[c, d]$, the largest (abs. max.) value is at S, not at P (still, the rel. max.), but the rel. min. value at Q is also the abs. min. value (smallest value) of the fn.

In (iii), the interval is $x > 0$ or $(0, \infty)$.
The rel. max value is at P + the rel. min. value at Q.
However, in this open interval, there is no abs. max. value and no abs. min. value.

Similar situations can occur for fns of 2 variables

Defn Let $f(x,y)$ be defined in the domain D in the xy-plane. If there is a point (x_0,y_0) in D such that

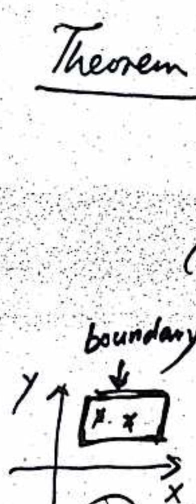
$$f(x_0,y_0) \geq f(x,y) \quad [\text{or } f(x_0,y_0) \leq f(x,y)]$$

for all points (x,y) in D, then $f(x_0,y_0)$ is said to be the Abs. Maximum Value [or Abs. Minimum Value] of f.

Theorem (a) If $f(x,y)$ is continuous in a closed domain R, then $f(x,y)$ must have the absolute maximum value and the absolute minimum value at some points of R.

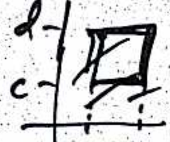
(b) If, in addition, f_x and f_y exist at all points of R, then the absolute max. (and absolute min.) of f must occur either at the critical pt. (x_0,y_0) [i.e. $f_x(x_0,y_0)=0$, $f_y(x_0,y_0)=0$] or at a point on the boundary of R.

boundary



$x^2+y^2<1$

Note: If R is not a closed domain, the fn may not have an abs. max or abs. min. Each situation is different.



$a \leq x \leq b$
~~$y \geq 0$~~
$c \leq y \leq d$

Ex. 1 A manufacturer finds that to produce x lamps of type 1 and y lamps of type 2, it will cost $C = 12x + 11y + 4xy$. However, the selling price of x lamps is $x(100-2x)$ and of y lamps is $y(125-3y)$.
How many lamps of each type should be produced to obtain the largest profit and what is the largest profit?

Soln Let the profit be

$$P = \text{Selling Price} - \text{Cost}$$

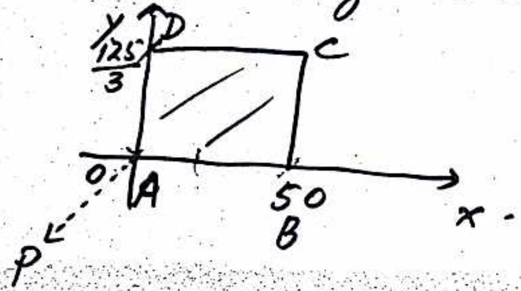
$$P = x(100-2x) + y(125-3y) - (12x + 11y + 4xy)$$

$$(1) \qquad P = 88x + 114y - 2x^2 - 3y^2 - 4xy.$$

Obviously, $x \geq 0$ & $y \geq 0$, as they are the no. of lamps; also $100 - 2x \geq 0$ and $125 - 3y \geq 0$, as they represent the selling price per unit lamp, i.e. $100 \geq 2x$ or $50 \geq x$ and $125 \geq 3y$ or $\frac{125}{3} \geq y$. Hence the domain of P is

(2). $\{(x,y) \mid \underline{0 \leq x \leq 50}$ and $0 \leq y \leq \frac{125}{3}\}$.

The domain is a rectangle shown, which is closed



with boundaries AB, BC, CD and DA.

By the above theorem, since P is a cts fn in the closed rectangle, P must attain an absolute max. and an absolute min. value. The absolute extremum must either be at some interior point of R or on the boundaries, since P_x & P_y exist.

$\therefore$ We consider the interior region first and then we consider the boundaries of R.

(i) In the interior of R: $0 < x < 50$, $0 < y < \frac{125}{3}$.

We proceed to determine the relative extremum of P.

$$\left.\begin{aligned} P_x &= 88 - 4x - 4y = 0 \\ P_y &= 114 - 6y - 4x = 0 \end{aligned}\right\} \text{ for critical points}$$

i.e. $\begin{cases} x + y = 22 \\ 2x + 3y = 57 \end{cases}$

Solving, $x = 9$ and $y = 13$.

$P_{xx} = -4$

$P_{yy} = -6$

$P_{xy} = -4$

$\therefore$ at the c.p. $(9,13)$, $P_{xx}P_{yy} - P_{xy}^2 = 24 - 16 = 8 > 0$

19.41 Hence, by the Second-Derivative Test, the fn P has a rel. max. value at the c.p. $(9,13)$.

$\therefore$ Rel. max. $P(x,y) = 88 \cdot 9 + 114 \cdot 13 - 2 \cdot 9^2 - 3 \cdot 13^2 - 4 \cdot 9 \cdot$

$= \underline{1137}$

(ii) Next, on each boundary of R:

(a) Along line AB: $0 \le x \le 50$, $y = 0$

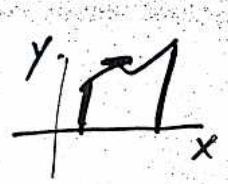
Here $P(x,0) = 88x - 2x^2$ (P is now regarded as a fn of x, on

$\therefore P' = 88 - 4x$

$= 0$ for critical pt. (fn of one variab

$\therefore x = 22$ is a c.p. + the corr. $P = 88 \cdot 22 - 2 \cdot 2$

$= \underline{968}$



Also, $P'' = -4$ (rel. max.)

$\therefore P$ has a rel. max at $x = 2$, and along line A

Rel. Max $P = \underline{968}$

The Rel. Max P is <u>also</u> the Abs. Max. <u>along line AB</u> since $\underline{P(0,0) = 0}$ and $\underline{P(50,0) < 0}$.

(b) Along the lines AD, BC + CD, we can similarly find the abs. max. of P.

<u>Along line AD</u>: $x = 0$, $0 \le y \le \frac{125}{3}$. Here

$P(0,y) = 114y - 3y^2$ (P is now only a fn of y.)

$\therefore P' = 114 - 6y$

$= 0$ for critical pt.

$\therefore y = \frac{114}{6} = 19$ is a c.p. + the corr.

$P = 114 \cdot 19 - 3 \cdot 19^2 = \underline{1083}$

Also $P'' = -6$ (rel. max.)

$\therefore P$ has a rel. max. at $y = 19$, of $\underline{1083}$.

This rel. max. is <u>also</u> the abs. Max. along line AD, since $P(0,0) = 0$ and $P(0, \frac{125}{3}) < 0$.

Along line BC: $x=50$, $0 \le y \le \frac{125}{3}$. Here

$$P(50,y) = 88\cdot 50 + 114y - 2(50)^2 - 3y^2 - 200y$$
$$= -86y - 3y^2 - 600$$
$$< 0. \quad \text{for } 0 \le y \le \tfrac{125}{3}.$$

Along line CD $\quad 0 \le x \le 50$, $y = \frac{125}{3}$. Here

$$P(x, \tfrac{125}{3}) = x(88-2x) - \frac{1375}{3} - \frac{500x}{3}.$$

(a) line AB. Comparing with $P(x,0) = x(88-2x)$, we see that

$$P(x, \tfrac{125}{3}) < P(x,0), \text{ for } 0 \le x \le 50.$$

$\therefore$ The abs. max of P along line CD

$<$ the abs. max of P along line AB.

Thus, the largest value of P in the interior and on the boundaries of R is. 1137 at the interior pt. (9,

and this must be the Abs. Max. Value of P.

To conclude, 9 lamps of type 1 and 13 lamps of type

should be produced for the largest profit of 1137.

Method of Lagrange Multipliers.

This method was used by Joseph L. Lagrange (1736-1813) to find the Rel. Extremum of a function, subject to some side conditions or constraints. For example:

| Find the minimum of
$$f(x,y,z)$$
subject to the constraint $g(x,y,z) = 0$. |

Lagrange introduced a variable λ, now called a Lagrange Multiplier. Then consider the new fn.

$$f(x,y,z) + \lambda g(x,y,z),$$

19.44 and solve the following equations to find the critical points:

$$\begin{cases} f_x + \lambda g_x = 0 \\ f_y + \lambda g_y = 0 \\ f_z + \lambda g_z = 0 \\ g(x,y,z) = 0 \end{cases}$$, and using

Then from the values of $f(x,y,z)$ at these critical poin[ts] the smallest value will be the rel. min., while the largest value will be the rel. max.

Ex. 1 Find the relative extrema of the fn.

$$f(x,y,z) = x^2 + y^2 + z^2 \qquad \geq 0$$

subject to $x+y+z=1$.

Soln Let $g(x,y,z) \equiv x+y+z-1 = 0$.

Then the critical pts are given by

$$f_x+\lambda g_x = 2x + \lambda = 0. \Rightarrow x = -\lambda/2$$
$$f_y+\lambda g_y = 2y + \lambda = 0 \qquad y = -\lambda/2$$
$$f_z+\lambda g_z = 2z + \lambda = 0 \qquad z = -\lambda/2.$$

Using these values in g,

$$-\frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} = 1$$

$$\therefore \frac{-3\lambda}{2} = 1$$

$$\lambda = \frac{-2}{3}.$$

$\therefore$ Critical pt. is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. & $f\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) = \frac{3}{9} = \frac{1}{3}$

To see if this c.p. gives a rel. max or min., we consider a neighbouring pt, say

Pg. 100 $\qquad x = \frac{1}{3} + h, \quad y = \frac{1}{3} + k, \quad z = \frac{1}{3} + l$

where $\underline{h + k + l = 0}$, (since $x + y + z = 1$).

Then at this neighbouring pt.

$$f\left(\tfrac{1}{3}+h, \tfrac{1}{3}+k + \tfrac{1}{3}+l\right) = \left(\tfrac{1}{3}+h\right)^2 + \left(\tfrac{1}{3}+k\right)^2 + \left(\tfrac{1}{3}+l\right)^2$$

$$= \frac{1}{9} + \frac{2}{3}h + h^2 + \frac{1}{9} + \frac{2}{3}k + k^2 + \frac{1}{9} + \frac{2}{3}l +$$

$h+k+l=0$ $\qquad$ $$= \frac{1}{3} + \frac{2}{3}(h+k+l) + h^2 + k^2 + l^2$$

$$= \frac{1}{3} + h^2 + k^2 + l^2.$$

$$> \frac{1}{3} = f\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$

This shows that

$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is the smallest value compared with neighbouring points. Hence f has a rel. min at $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and the value $f = \frac{1}{3}$ is $\therefore$ a <u>minimum</u>.

<u>Ex. 2.</u> Find the max. or min. of

$$f = xyz$$

under the condition $\quad x + y + z = 1$

<u>Soln</u> Let $\quad g \equiv x + y + z - 1 = 0 \qquad (1)$

Then

$$f_x + \lambda g_x = yz + \lambda = 0 \qquad (2)$$

$$f_y + \lambda g_y = xz + \lambda = 0 \qquad (3)$$

$$f_z + \lambda g_z = xy + \lambda = 0 \qquad (4)$$

Mult. (2) by x, $\qquad xyz + \lambda x = 0$

" (3) by y, $\qquad xyz + \lambda y = 0$

" (4) by z; $\qquad xyz + \lambda z = 0$

Adding these 3 eqs., we get

$$3xyz + \lambda(x+y+z) = 0$$

Using (1), $3xyz + \lambda \cdot 1 = 0$ (5)

From (2) & (5) $3x = 1$

or $x = \frac{1}{3}$

From (3) & (5) $y = \frac{1}{3}$

From (4) & (5) $z = \frac{1}{3}$.

$\therefore$ The critical pt. is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. & the corresponding value $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \underline{\frac{1}{27}}$.

To see if this pt. is a rel. max or min, we let

$$x = \tfrac{1}{3} + h, \quad y = \tfrac{1}{3} + k, \quad y = \tfrac{1}{3} + l$$

where h, k, l are small. Since $x+y+z=1$, we see that $h+k+l=0$.

Now $f = (\frac{1}{3}+h)(\frac{1}{3}+k)(\frac{1}{3}+l)$

$$= \frac{1}{27} + \frac{1}{9}\cancel{(h+k+l)} + \frac{1}{3}(hk+kl+lh) + hkl.$$

We can delete hkl, as h, k, l are small, so that

$$f = \frac{1}{27} + \frac{1}{3}\left\{\frac{1}{2}(h+k+l)^2 - \frac{1}{2}(h^2+k^2+l^2)\right\}$$

$$= \frac{1}{27} - \frac{1}{6}(h^2+k^2+l^2), \text{ since } h+k+l=0$$

$$< \frac{1}{27} = f(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})$$

Hence f has its relative largest value at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and the value $\frac{1}{27}$ is $\therefore$ a maximum value of f.