1. If w and z are complex numbers, show that $\bar{w}\bar{z} = \overline{wz}$.

2. Derive the addition law for sine and cosine using Euler's formula.

3. We can use this to find the roots of a cubic equation as follows.

   (a) First show that since

   $$4\cos^3\phi - 3\cos\phi = \cos 3\phi,$$

   and k is any other number, it follows that

   $$4(k\cos\phi)^3 - 3k^2(k\cos\phi) = k^3\cos 3\phi$$

   and hence that $k\cos\phi$ is a root of the cubic equation

   $$x^3 - \frac{3k^2}{4}x - \frac{k^3\cos 3\phi}{4} = 0.$$

   (b) Show that the other roots of this equation are $k\cos(\phi + 2\pi/3)$ and $k\cos(\phi + 4\pi/3)$.

   (c) The equation constructed is the same as $x^3 + 3Hx + G = 0$ if we choose k and ϕ so that

   $$-\frac{1}{4}k^2 = H, \qquad -\frac{1}{4}k^3\cos 3\phi = G.$$

   The choice can be made with k real if and only if $H \leq 0$, and with ϕ real if and only if $G = H = 0$ or else $H < 0$ and G is real with $G^2 = -4H^3\cos^2 3\phi \leq -4H^3$. From algebra this corresponds to the case where all three roots are real.

   (d) By changing the variable $y = x + a$, write the cubic

   $$8y^3 + 24y^2 + 6y - 1 = 0$$

   in the depressed form

   $$x^3 - \frac{9}{4}x + \frac{9}{8} = 0.$$

(e) Show that the depressed cubic has roots

$$\sqrt{3}\cos\frac{5\pi}{18}, \qquad \sqrt{3}\cos\frac{17\pi}{18}, \qquad \sqrt{3}\cos\frac{29\pi}{18},$$

and that the roots of the original cubic are obtained by adding 1 to each of these three numbers.

It is interesting to note that for the case $G^2 + 4H^3 > 0$ we can do exactly the same trick by using the addition formula for the hyperbolic cosine function $\cosh\phi$

4. Find the eigenvalues of the following matrices

$$A = \begin{pmatrix} -1 & 1 & 1 \\ -3 & 3 & 1 \\ -4 & 3 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

5. Find eigenvectors for the eigenvalues you found in the previous exercise.