The University of Calgary
Department of Mathematics and Statistics
MATH 349
Handout # 3 Solution

For 1a)

$\sum\limits_{n=3}^{\infty} |...| = \sum\limits_{n=3}^{\infty} \dfrac{1}{n(\ln n)}$ is divergent by Integral test:

$f(x) = \dfrac{1}{x \ln x}$ is positive,continous and decreasing for $x > 1$

since $x \ln x$ is the product of two positive and increasing functions,

The integral is divergent,since

$\int\limits_{3}^{\infty} \dfrac{1}{x(\ln x)} dx = \lim\limits_{x\to\infty} \ln(\ln x) - \ln\ln 3 = \infty$

so the original series is absolutely divergent.

For 1b)

it is conditionally convergent since $a_n = \dfrac{1}{n(\ln n)} \to 0 \left(\frac{1}{\infty}\right)$

and the sequence is decreasing — see above f is decreasing function

For 2a)

the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{\ln(n+1)}$ is divergent by Comparison Test since $0 < \ln x < x$ for $x > 1$,

$\ln(n+1) < n+1 \qquad \dfrac{1}{\ln(n+1)} > \dfrac{1}{n+1}$

and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n+1} = \sum\limits_{k=2}^{\infty} \dfrac{1}{k}$ which is divergent harmonic series (by Integral Test).

For 2 b)

the series is conditionally convergent by Alternating Test since the sequence

$a_n = \dfrac{1}{\ln(n+1)}$ has positive terms,limit 0 $\left(\frac{1}{\infty}\right)$and is decreasing

since $\ln x$ is increasing,positive for $x > 1$, and "$\frac{1}{\text{incr,pos}}$" = decr.

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For 3a)

investigate $\sum\limits_{n=2}^{\infty} (\frac{1}{n} - \frac{1}{n!})$ since $a_n = \left(\frac{1}{n} - \frac{1}{n!}\right) > 0$ for $n \geq 3$ We can split into two series

harmonic series $\sum\limits_{n=2}^{\infty} \frac{1}{n}$ which is divergent and the series $\sum\limits_{n=2}^{\infty} \frac{1}{n!}$ which is convergent

by Ratio Test: $0 < \frac{1}{(n+1)!} \cdot n! = \frac{1}{n+1} \to 0 < 1$,

so together the series $\sum\limits_{n=2}^{\infty} (\frac{1}{n} - \frac{1}{n!})$ is divergent.The original series is absolutely divergent.

Also by Comparison Test $\quad$ for $n \geq 3 \quad n! \geq 2n$ so $\frac{1}{n!} \leq \frac{1}{2n}$ and

$a_n \geq \frac{1}{2n}$ for $n > 2$.

For 3b)

we can separate again since $\sum\limits_{n=2}^{\infty} (-1)^n \frac{1}{n}$ is conditionally convergent by Alternating Test:

the sequence $\left\{\frac{1}{n}\right\}$ is decreasing,with positive terms and limit 0.

And the second series $\sum\limits_{n=2}^{\infty}(-1)^n\frac{1}{n!}$ is absolutely convergent from above ,so together the original series is conditionally convergent.

Also directly by Alternating test:

the sequence a_n from above has limit 0 ,so we have to show that it is decreasing:

$a_{n+1} < a_n \qquad \frac{1}{n+1} - \frac{1}{(n+1)!} < \frac{1}{n} - \frac{1}{n!}$, multiply both sides by $(n+1)!$:

$n! - 1 < (n+1)(n-1)! - (n+1)$, so $n! < n(n-1)! + (n-1)! - n, 0 < (n-1)! - n$

finally $n < (n-1)(n-2) < (n-1)!$ which is true for $n \geq 4$

For 4a)

the centre is $c = -1$ and $a_n = \dfrac{n!}{4^n}$, so $0 < \dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{4^{n+1}} \cdot \dfrac{4^n}{n!} = \dfrac{n+1}{4} \to \infty$,

so $R = \frac{1}{L} = 0$ and the series converges ONLY for $x = -1$.

For 4b)

the centre is $c = \frac{1}{2}$ and $a_n = \dfrac{n! \cdot 2^n}{(2n)!}$, so $0 < \dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!2^{n+1}}{(2n+2)!} \cdot \dfrac{(2n)!}{n!2^n} =$

$= \dfrac{(n+1)2}{(2n+2)(2n+1)} = \dfrac{1}{2n+1} \to 0$ so $R = \frac{1}{L} = +\infty$, and the interval is $(-\infty, +\infty)$.

For 5)

The answer must be in the form $\sum\limits_{n=1}^{\infty} a_n (x-1)^n$ .We know that for geometric series

$\sum\limits_{n=0}^{\infty}(-1)^n r^n = \dfrac{1}{1+r}$ for any $-1 < r < 1$.

So first $\qquad \dfrac{1}{x+1} = \dfrac{1}{(x-1)+2} = \frac{1}{2} \cdot \dfrac{1}{1+\frac{x-1}{2}} \qquad \left(r = \frac{x-1}{2}\right)$

$\dfrac{1}{x+1} = \frac{1}{2} \cdot \sum\limits_{n=0}^{\infty}(-1)^n \dfrac{(x-1)^n}{2^n}$ for $-1 < x < 3$ (from $-1 < \dfrac{x-1}{2} < 1$).

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Differentiate both sides: $\qquad \dfrac{-1}{(x+1)^2} = \frac{1}{2}\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{2^n} \cdot n(x-1)^{n-1}, (n-1 = k)$

$\dfrac{1}{(x+1)^2} = \sum\limits_{k=0}^{\infty}\dfrac{(-1)^k \cdot (k+1)}{2^{k+2}}(x-1)^k$ for $-1 < x < 3$.

F

For 6a)

$\sum\limits_{n=1}^{\infty}\dfrac{(4x-1)^n}{n^n} = \sum\limits_{n=1}^{\infty}\dfrac{4^n(x-\frac{1}{4})^n}{n^n} \qquad$ the centre is $c = \frac{1}{4}$ and $a_n = \dfrac{4^n}{n^n}$.

Since $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{4^{n+1}}{(n+1)^{n+1}} \cdot \dfrac{n^n}{4^n} = \dfrac{4}{n+1} \cdot \left(\dfrac{n}{n+1}\right)^n \to L = 0 \cdot \frac{1}{e} = 0$,

$R = \frac{1}{L} = +\infty$, the interval is $(-\infty, +\infty)$.

Also by Root Test $(|a_n|)^{\frac{1}{n}} = \frac{4}{n} \to 0$.

For 6b)

the centre is $c = 4$ and $a_n = (-1)^n \dfrac{n}{2^n}$,since $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{n+1}{2^{n+1}} \cdot \dfrac{2^n}{n} \to \dfrac{1}{2}$

the radius is $R = 2$ and series is absolutely convergent on $(2, 6)$.

Now,for $x = 2$

we have to investigate $\sum_{n=1}^{\infty} n = +\infty$ and for $x = 6$ the series $\sum_{n=1}^{\infty} (-1)^n n$ which is also divergent since ther limit of the n-th term is NOT 0.
So the interval is open $(2, 6)$.

For 7)

The answer must be in the form $\sum_{n=0}^{\infty} a_n (x+1)^n$ .Rewrite $\ln(2-x) = \ln(3-(x+1)) =$

$= \ln\left[3(1-\frac{x+1}{3})\right] = \ln 3 + \ln(1-\frac{x+1}{3}) = \ln 3 - \sum_{n=1}^{\infty} \frac{1}{n3^n} (x+1)^n$

for $-1 \leq \frac{x+1}{3} < 1 \qquad -3 \leq x+1 < 3 \qquad -4 \leq x < 2, x \in [-4, 2)$

using $\ln(1-s) = -\sum_{n=1}^{\infty} \frac{1}{n} s^n$ for $s \in [-1, 1)$

For 8)

$$\frac{1}{1-2x} = \frac{1}{1-2(x+4-4)} = \frac{1}{9-2(x+4)} = \frac{1}{9} \cdot \frac{1}{1-\frac{2}{9}(x+4)}$$

now,using $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ for $-1 < r < 1$,where $r = \frac{2(x+4)}{9}$ we get

$$\frac{1}{1-2x} = \frac{1}{9} \sum_{n=0}^{\infty} \frac{2^n}{9^n}(x+4)^n = \sum_{n=0}^{\infty} \frac{2^n}{9^{n+1}}(x+4)^n, \text{ so } a_n = \frac{2^n}{9^{n+1}}$$

Now , to find the interval ,solve for x: $\qquad -1 < \frac{2(x+4)}{9} < 1$

$-9 < 2x+8 < 9 \qquad \frac{-17}{2} < x < \frac{1}{2}$.