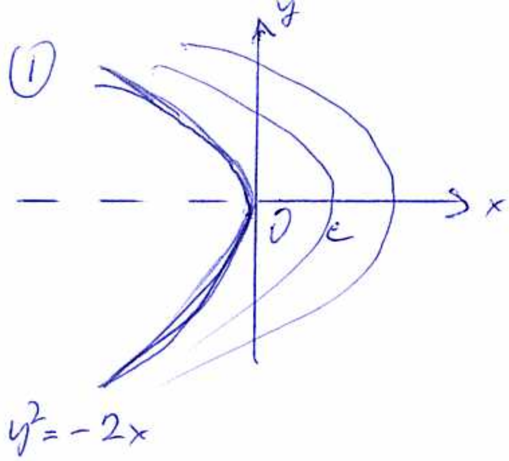
MATH 349
Handout #5 Solution

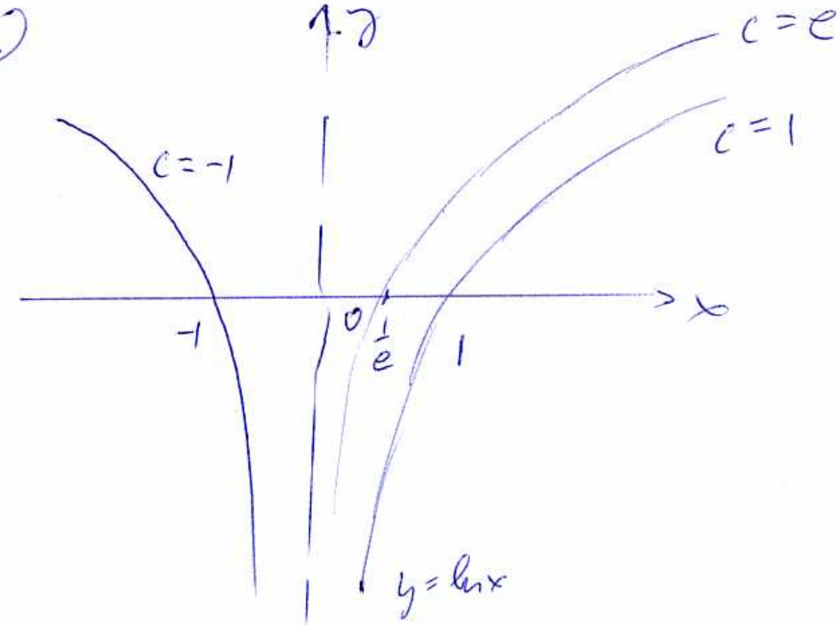
For 1)
$f(x, y) = \sqrt{2x + y^2}$
for the domain $2x + y^2 \geq 0 \qquad y^2 \geq -2x \qquad$ any $x > 0$ will do,
generally the region right of the parabola $y^2 = -2x$, $x \leq 0$
level curves: $c = 0 \qquad 0 = 2x + y^2$.... parabola above;
$c < 0$.... NO curves; $c > 0$...parabolas shifted to the right
$y^2 = c^2 - 2x$ ($y = \pm\sqrt{c^2 - 2x}$)
Partials:
$f_x = \dfrac{2}{2\sqrt{2x + y^2}} = \dfrac{1}{\sqrt{2x + y^2}}$; $f_y = \dfrac{2y}{2\sqrt{2x + y^2}} = \dfrac{y}{\sqrt{2x + y^2}}$
obviously $\qquad y\ f_x = f_y$ in the domain except on $y^2 = -2x$.
For 2)
For $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2 + 2y^2}$
define $g(x, y) = \dfrac{x^3}{x^2 + 2y^2}$ then $g(0, y) = 0$ for $y \neq 0$ and for $x \neq 0$
$g(x, 0) = \frac{x^3}{x^2} = x \to 0$ as $x \to 0$; along any line $y = mx$ for $x \neq 0$
$g(x, mx) = \dfrac{x^3}{x^2 + 2m^2x^2} = \dfrac{x}{1 + 2m^2} \to 0$ as $x \to 0$.
Thus limit could be 0. Now,we have to prove it
$|g(x, y) - 0| = \left|\dfrac{x^3}{x^2 + 2y^2}\right| = |x| \dfrac{x^2}{x^2 + 2y^2} \leq |x| \dfrac{x^2 + 2y^2}{x^2 + 2y^2} = |x| \to 0$
as $x \to 0$ therefore $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2 + 2y^2} = 0$ by Squeeze Theorem.
For 3)
For $f(x, y) = \dfrac{e^y}{x}$ the domain is $x \neq 0$,so $D = \mathbf{R}^2 - \{y - axis\}$
level curves: $c = 0$,NO curve since $e^y > 0$ always
for $c \neq 0 \qquad cx = e^y, \ln cx = y$,for $cx > 0$,and $y = \ln|x| + \ln|c|$
so for $c = 1, y = \ln x, x > 0$; for $c = -1, y = \ln(-x), x < 0$,
and for $c = e, y = \ln x + 1, x > 0$
For the derivative $f_x = -\dfrac{e^y}{x^2}$,and $(f_x)_y = -\dfrac{e^y}{x^2} = f_{xy}$
since all functions are continuous in $D \qquad f_{xy} = f_{yx}$.
For 4)
for $f(x, y) = \left(yx^2 + y^2 + \frac{1}{4}x^4\right)^{-\frac{1}{2}}$
using Chain Rule $f_x = -\frac{1}{2}\left(yx^2 + y^2 + \frac{1}{4}x^4\right)^{-\frac{3}{2}} \cdot (2xy + x^3)$ and
$f_y = -\frac{1}{2}\left(yx^2 + y^2 + \frac{1}{4}x^4\right)^{-\frac{3}{2}} \cdot (x^2 + 2y)$,so $\qquad xf_y = f_x$.
For the domain: $yx^2 + y^2 + \frac{1}{4}x^4 > 0, \left(y + \frac{1}{2}x^2\right)^2 > 0$,
always except when $y = -\frac{1}{2}x^2$
Also we can simplified $f(x, y) = \left|y + \frac{1}{2}x^2\right|^{-1}$,so $f_x = \dfrac{-x}{(y + \frac{1}{2}x^2)^2} sgn\left(y + \frac{1}{2}x^2\right)$

(1)

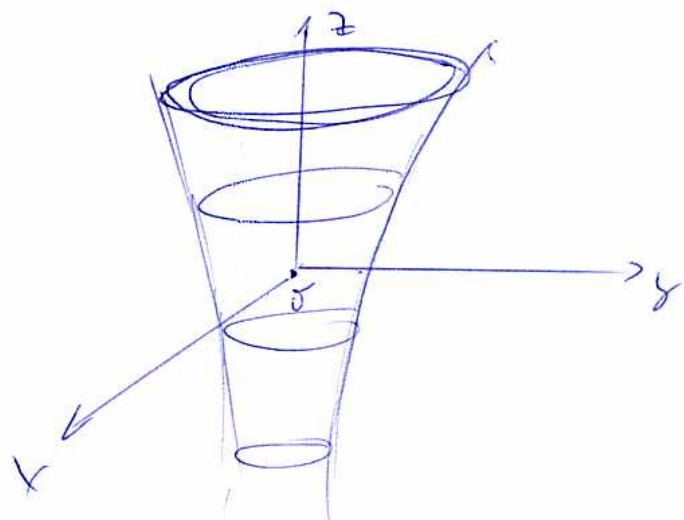


y
0
e
x
$y^2 = -2x$

(3)

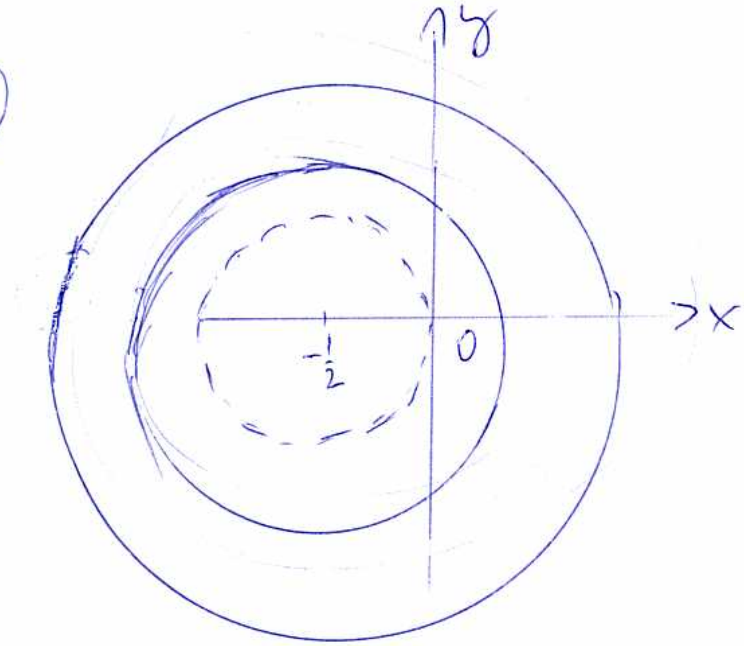


y
c = e
c = 1
c = -1
-1
0
$\frac{1}{e}$
1
x
$y = \ln x$



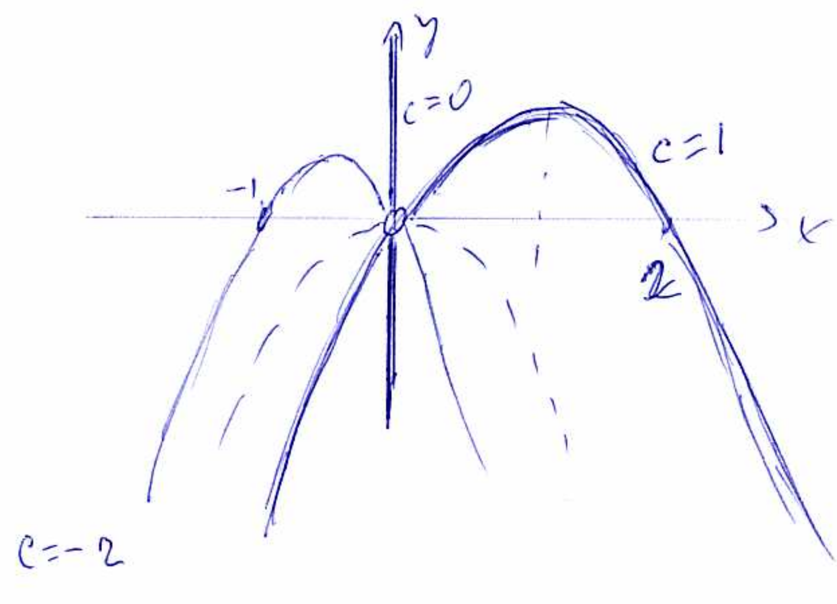
z
0
y
x

(5)



y
x
$-\frac{1}{2}$
0

(8)



y
c = 0
c = 1
-1
2
x
c = -2

and $\qquad f_y = \dfrac{-1}{(y+\frac{1}{2}x^2)^2} sgn\left(y+\frac{1}{2}x^2\right)$,if $y+\frac{1}{2}x^2 \neq 0$.

For 5)

For $f(x,y) = \ln(x^2+y^2+x)$

for domain solve :$x^2+y^2+x>0$ complete the square: $\left(x+\frac{1}{2}\right)^2+y^2>\frac{1}{4}$

so the domain is outside the circle with the centre at $(-\frac{1}{2},0)$ and $r=\frac{1}{2}$.

For the level curves: $e^c = x^2+y^2+x$ so as above :

$\left(x+\frac{1}{2}\right)^2+y^2 = e^c+\frac{1}{4}$

so all are circles with the same centre $(-\frac{1}{2},0)$ and $r=\sqrt{e^c+\frac{1}{4}}$,

particularly if $c=0 \qquad r=\frac{\sqrt{5}}{2}; c=\ln 2 \qquad r=\frac{3}{2};$

$c=-\ln 2=\ln\frac{1}{2} \qquad r=\frac{\sqrt{3}}{2}...$

Notice that cross-section $x=0 \qquad z=2\ln|y|$

so we can sketch the graph of f :

For 6)

define $g(x,y)=\dfrac{xy+y}{(x+1)^2+y^2}=\dfrac{(x+1)y}{(x+1)^2+y^2}$ for $(x,y)\neq(-1,0)$

then $g(-1,y)=\frac{0}{y^2}=0$ for $y\neq 0$ and $g(x,0)=0$ for $x\neq -1$

but for the line$y=x+1$passing through the point $(-1,0)$

$g(x,x+1)=\dfrac{(x+1)^2}{2(x+1)^2}=\dfrac{1}{2}$ for $x\neq -1$ so the limit DNE.

For 7)

$f(x,y)=\arctan\dfrac{x}{y}$ for $y\neq 0$

$f_x=\dfrac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}=\dfrac{y}{x^2+y^2}$ and $f_y=\dfrac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\dfrac{-x}{y^2}=\dfrac{-x}{x^2+y^2}$

so $\qquad y\ f_x - x\ f_y = \dfrac{y^2+x^2}{x^2+y^2}=1$ for any x and $y\neq 0$.

For 8)

For $f(x,y)=\dfrac{2x}{x^2+y}$ the domain is

$D=\{y\neq -x^2\}$ ...$xy-$plane except the parabola $y=-x^2$

level curves $c=0 \Longrightarrow 2x=0, x=0$...$y-$axis except the origin

for $c=1$: $\qquad 1=\dfrac{2x}{x^2+y}, \quad x^2+y=2x, \quad y=-x^2+2x=-x(x-2)$

a parabola open down,vertex $V(1,1)$ without the origin

for $c=-2$: $\quad -2=\dfrac{2x}{x^2+y}, \quad -2x^2-2y=2x, \quad y=-x(1+x)$

a parabola open down ,vertex at $V(-\frac{1}{2},\frac{1}{4})$,without the origin

general level curves are shifted parabolas open down without the origin,

passing through the origin $\qquad$ now,partials

$f_x(x,y)=2\cdot\dfrac{x^2+y-2x^2}{(x^2+y)^2}=\dfrac{2(y-x^2)}{(x^2+y)^2}$ at $x=-1, y=1 \quad f_x(-1,1)=0$

$f_y(x,y)=2x\cdot(-1)(x^2+y)^{-2}\cdot 1=\dfrac{-2x}{(x^2+y)^2}$ at $x=-1,y=1 \quad f_y(-1,1)=\frac{2}{4}=\frac{1}{2}$

so a normal vector to the tangent plane is $(f_x,f_y,-1)=\left(0,\frac{1}{2},-1\right)$

or $\overrightarrow{n} = (0, 1, -2)$ an equation is $\quad y - 2z = d$
for d we need the point $z_0 = f(-1, 1) = \frac{-2}{2} = -1 \quad P(-1, -1, -1)$
and $1 + 2 = 3 = d$,so together $\quad y - 2z = 3.$

For 9)

define $g(x, y) = \dfrac{xy - x}{3x^2 + 2(y-1)^4}$ for $(x, y) \neq (0, 1)$
then for $y \neq 1 \quad g(0, y) = 0$ and for $x \neq 0 \quad g(x, 1) = 0,$
try a line through that point $y = mx + 1$
for $x \neq 0 \quad f(x, mx + 1) = \dfrac{mx^2}{3x^2 + 2m^4x^4} = \dfrac{m}{3 + 2m^4x^2} \to \frac{m}{3} \neq 0$
for any $m \neq 0 \qquad$ (as $x \to 0$) $\qquad$ Therefore the limit DNE.