The University of Calgary
Department of Mathematics and Statistics
MATH 349 Handout # 5

Solutions **For 1 a)**

For $f(x,y) = \dfrac{xy}{\sqrt{1+x^2}}$ $\quad f_y = \dfrac{x}{\sqrt{1+x^2}}$ and

$$f_x = y \cdot \frac{\sqrt{1+x^2} - x\frac{2x}{2\sqrt{1+x^2}}}{(1+x^2)} = y\frac{(1+x^2)-x^2}{\sqrt{1+x^2}\,(1+x^2)} = \frac{y}{(1+x^2)^{\frac{3}{2}}}$$

For 1b)

for $x = \cos(\pi st)$ $\quad y = \sin\dfrac{\pi s}{t}$ $\quad \dfrac{\partial x}{\partial s} = -\sin(\pi st)\cdot \pi t$ $\quad \dfrac{\partial y}{\partial s} = \cos\dfrac{\pi s}{t}\cdot\dfrac{\pi}{t}$

and $\dfrac{\partial x}{\partial s}(0,-1) = 0$ $\quad \dfrac{\partial y}{\partial s}(0,-1) = -\pi$

for $s = 0$ and $t = -1$ $\quad x = \cos(0) = 1, y = \sin 0 = 0$

and $\nabla f(1,0) = \left(0, \dfrac{1}{\sqrt{2}}\right)$

$\frac{\partial}{\partial s}h(0,-1) = f_x\frac{\partial x}{\partial s} + f_y\frac{\partial y}{\partial s} = \nabla f(1,0) \bullet \left(\frac{\partial x}{\partial s}(0,-1), \frac{\partial y}{\partial s}(0,-1)\right) =$

$= \left(0, \frac{1}{\sqrt{2}}\right) \bullet (0,-\pi) = \dfrac{-\pi}{\sqrt{2}}$

$\dfrac{\partial x}{\partial t} = -\sin(\pi st)\cdot\pi s$ $\quad \dfrac{\partial y}{\partial t} = \cos\frac{\pi s}{t}\cdot\dfrac{-\pi s}{t^2}$

and $\dfrac{\partial x}{\partial t}(0,-1) = 0$ $\quad \dfrac{\partial y}{\partial t}(0,-1) = 0$

$\frac{\partial}{\partial t}h(0,-1) = f_x\frac{\partial x}{\partial t} + f_y\frac{\partial y}{\partial t} = \left(0, \frac{1}{\sqrt{2}}\right) \bullet (0,0) = 0$

OR by Chain Rule $\quad Dh = Df\,D\mathbf{g}$ $\quad$ matrix multiplication where

$\mathbf{Dg} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$ at $s = 0, t = -1$ $\quad \mathbf{Dg}(0,-1) = \begin{bmatrix} 0 & 0 \\ -\pi & 0 \end{bmatrix}$

so $Dh = \nabla h = \begin{bmatrix} 0 & \dfrac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -\pi & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{-\pi}{\sqrt{2}} & 0 \end{bmatrix}$

For 2)

for $f(x,y) = \ln(x+y^2)$ and $x_0 = 0, y_0 = -1$ $\quad z_0 = f(0,-1) = \ln 1 = 0$

partials $f_x = \dfrac{1}{x+y^2}$ $\quad f_y = \dfrac{2y}{x+y^2}$ $\quad$ and $A = f_x(0,-1) = 1, B = f_y(0,-1) = -2$

so tangent plane $\quad z = z_0 + A(x-x_0) + B(y-y_0)$ $\quad z = 0 + (x-0) - 2(y+1) = x - 2y - 2$

Or the point on the graph is $P(0,-1,0)$ and $\mathbf{n} = (\nabla f, -1) = (1,-2,-1)$

so $x - 2y - z = d$ and from P $\quad 0 + 2 - 0 = d$ thus $x - 2y - z = 2$.

For 3)

for $f(x,y) = e^{\sqrt{\frac{y}{x}}}$ the domain is $\frac{y}{x} \geq 0$ i.e.$\{y \geq 0, x > 0\} \cup \{y \leq 0, x < 0\}$

$\left(-\frac{\sqrt{y}}{2}x^{-\frac{3}{2}}\right) = -\frac{1}{2x}\sqrt{\frac{y}{x}}\;e^{\sqrt{\frac{y}{x}}}$

$f_{xx} = e^{\sqrt{\frac{y}{x}}}\left[\left(-\frac{\sqrt{y}}{2}x^{-\frac{3}{2}}\right)^2 + \frac{3}{4}\sqrt{y}x^{-\frac{5}{2}}\right] = e^{\sqrt{\frac{y}{x}}}\left[\frac{y}{4x^3} + \frac{3}{4x^2}\sqrt{\frac{y}{x}}\right]$

$f_{xy} = -\frac{1}{2}x^{-\frac{3}{2}}e^{\sqrt{\frac{y}{x}}}\left[\frac{1}{2\sqrt{y}} + \frac{\sqrt{y}}{2\sqrt{y}\sqrt{x}}\right] = -\frac{1}{4}e^{\sqrt{\frac{y}{x}}}\cdot\left(\dfrac{1}{x\sqrt{yx}} + \dfrac{1}{x^2}\right), y \neq 0$

the middle steps are true only if both x, y are positive but the final expresions are valid even if both are negative.

OR $f_x = e^{\sqrt{\frac{y}{x}}} \cdot \frac{1}{2}\sqrt{\frac{x}{y}} \cdot \frac{-y}{x^2}$ by Chain Rule

For 4a)

$\nabla f = gradf = (f_x, f_y, f_z)$ where

$f_x = \sqrt{2}\cos(\pi xy + x\ln z)\,[\pi y + \ln z]$ $\quad$ $f_y = \sqrt{2}\cos(\pi xy + x\ln z)\,[\pi x]$

$f_z = \sqrt{2}\cos(\pi xy + x\ln z)\left[\frac{x}{z}\right]$

For 4b)

$\mathbf{g}'(t) = (g_1'(t), g_2'(t), g_3'(t)) = \left(-\frac{1}{t^2}, \frac{1}{t^2}, \frac{1}{2}\right)$ for $t \neq 0$

OR

$$D\mathbf{g} = \begin{bmatrix} g_1' \\ g_2' \\ g_3' \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{t^2} \\ \dfrac{1}{t^2} \\ \dfrac{1}{2} \end{bmatrix}$$, where $g_1(t) = \frac{1}{t}, g_2(t) = -\frac{1}{t}$, and $g_3(t) = \frac{t}{2}$

For 4c)

if $t = 2$ then $x = \frac{1}{2}, y = \frac{-1}{2}$ and $z = 1$ $\quad$ $\sqrt{2}\cos(\pi xy + x\ln z) = \sqrt{2}\cos\frac{-\pi}{4} = 1$

and by Chain Rule $\quad$ $Dh = Df\ D\mathbf{g}$

$Df = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} = \begin{bmatrix} -\dfrac{\pi}{2} & \dfrac{\pi}{2} & \dfrac{1}{2} \end{bmatrix}$ at $(\frac{1}{2}, -\frac{1}{2}, 1)$ and $\mathbf{g}'(2) = \left(\dfrac{-1}{4}, \dfrac{1}{4}, \dfrac{1}{2}\right)$

$$h'(2) = Df(\tfrac{1}{2}, -\tfrac{1}{2}, 1)\mathbf{Dg}(2) = \begin{bmatrix} -\dfrac{\pi}{2} & \dfrac{\pi}{2} & \dfrac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{-1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = \frac{\pi}{8} + \frac{\pi}{8} + \frac{1}{4} = \frac{\pi + 1}{4}.$$

OR $h'(2) = f_x x' + f_y y' + f_z z' = \nabla f(\frac{1}{2}, -\frac{1}{2}, 1) \bullet (g_1'(2), g_2'(2), g_3'(2)) =$

$= \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{1}{2}\right) \bullet \left(\dfrac{-1}{4}, \dfrac{1}{4}, \dfrac{1}{2}\right) = \dfrac{\pi + 1}{4}$

For 5)

Simplify the function first $\quad$ $f(x, y) = \ln\frac{x^2+y^2}{xy} = \ln(x^2 + y^2) - \ln x - \ln y$

for $x > 0, y > 0$ so

partials are $f_x = \dfrac{2x}{x^2 + y^2} - \dfrac{1}{x}$ $\quad$ $f_y = \dfrac{2y}{x^2 + y^2} - \dfrac{1}{y}$

and $f_x(2, 1) = \frac{4}{5} - \frac{1}{2} = \dfrac{3}{10}$ $\quad$ $f_y(2, 1) = \frac{2}{5} - 1 = \dfrac{-3}{5}$

Now, rate of change means the directional derivative

Since partials are continuous in the domain

$D_\nu f(2, 1) = \nabla f(2, 1) \bullet (\nu_1, \nu_2) = \frac{3}{10}$, where $\nu_1^2 + \nu_2^2 = 1$,

so $\frac{3}{10}\nu_1 - \frac{3}{5}\nu_2 = \frac{3}{10}$ $\quad$ $\nu_1 - 2\nu_2 = 1$

one solution is $\nu_1 = 1, \nu_2 = 0$,

generally $\nu_1 = 1 + 2\nu_2 \Longrightarrow \nu_1^2 + \nu_2^2 = 5\nu_2^2 + 4\nu_2 + 1 = 1$,

$\nu_2(5\nu_2 + 4) = 0$ thus another solution is $\nu_2 = -\frac{4}{5}$ and $\nu_1 = 1 - \frac{8}{5} = -\frac{3}{5}$

Together $\boldsymbol{\nu} = (1, 0)$ or $\boldsymbol{\nu} = \frac{1}{5}(-3, -4)$

Maximum rate is always equal to $\|\nabla f\| = \left\|\left(\dfrac{3}{10}, -\dfrac{3}{5}\right)\right\| = \dfrac{3\sqrt{5}}{10}$.

For 6)

$\nabla f = (f_x, f_y, f_z) = (2xze^y + z^2, x^2ze^y, x^2e^y + 2xz)$ at P
$\nabla f(P) = (12, 4, 6) = 2\,(6, 2, 3,)\qquad \|\nabla f\| = 2\sqrt{49} = 14$
the direction is $\mathbf{u} = -\dfrac{\nabla f}{\|\nabla f\|} = -\dfrac{1}{7}\,(6, 2, 3)$ and rate is $\|\nabla f\| = 14$.

For 7a)

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{2x^2 + y^2}$ does not exists,for sure it is NOT equal to 0
since for $x \neq 0 \qquad f(x, x) = \dfrac{1}{3}$ so f is discontinous at $(0, 0)$;

For 7b)

by definition $\qquad f_x\,(0, 0) = \lim\limits_{x\to 0} \dfrac{f(x, 0) - 0}{x} = 0$

$f_y\,(0, 0) = \lim\limits_{y\to 0} \dfrac{f(0, y) - 0}{y} = 0$ so the gradient exists and $\nabla f\ (0, 0) = (0, 0)$.

For 7c)

for the directional derivative we have to use the definition
since f is discont.at $(0, 0)$ thus partials cannot be cont.at$(0, 0)$
unit vector in the direction of the line $y = x \qquad x = t, y = t \qquad$ is $\mathbf{u} = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

$$D_u f(0, 0) = \lim_{t\to 0} \frac{f(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}) - 0}{t} = \lim_{t\to 0} \frac{1}{t} \frac{\frac{1}{2}t^2}{\frac{3}{2}t^2} = \lim_{t\to 0} \frac{1}{3t} \text{does not exists}$$

OR from a)
since $f(x, x) = \dfrac{1}{3}$ for $x \neq 0$ and $f(0, 0) = 0$ f is not continous along that line
so it cannot be differentiable along that line.

For 7d)

for any other point we can use the theorem since partials are continuous
$D_u f = \nabla f \bullet \mathbf{u}$

first $\qquad f_x = y\dfrac{2x^2 + y^2 - 4x^2}{(2x^2 + y^2)^2} = y\dfrac{y^2 - 2x^2}{(2x^2 + y^2)^2}$

and $\qquad f_y = x\dfrac{2x^2 + y^2 - 2y^2}{(2x^2 + y^2)^2} = x\dfrac{2x^2 - y^2}{(2x^2 + y^2)^2}$ so at the given point

$\nabla f\,(-1, -1) = \left(\dfrac{1}{9}, -\dfrac{1}{9}\right)$ and $D_u f\,(-1, -1) = \dfrac{1}{9\sqrt{2}}\,(-1, -1) \bullet (1, 1) = 0$