MATH 349
Midterm Handout-Solution

1. Determine if the indicated sequence is bounded, monotonic, and convergent

a) $a_n = \dfrac{\ell n(n+3)}{n+3}$ b) $b_n = \dfrac{n^n}{n!}$.

For a)

$$\lim_{n\to\infty} a_n = \text{"}\frac{\infty}{\infty}\text{"} L'H.R. = \lim_{x\to\infty} \frac{\frac{1}{x+3}}{1} = \text{"}\frac{1}{\infty}\text{"} = 0$$

so the sequence is convergent and thus bounded.

For monotonicity ,define $f(x) = \dfrac{\ln x}{x}$

$f'(x) = \dfrac{\frac{1}{x}\cdot x - \ln x}{x^2} = \dfrac{1-\ln x}{x^2} < 0$ for $\ln x > 1, x > e$,

thus the sequence is decreasing for $x = n+3 \geq 3$,$n \geq 1$

and an lower bound is 0,an upper bound is $a_1 = \frac{\ln 4}{4}$.

For b)

$b_n = \dfrac{n^n}{n!} = \dfrac{n\cdot n\cdot \ldots \cdot n}{n(n-1)\ldots.2\cdot 1} > n$ so $\lim b_n = +\infty$

it is possible to investigate the series $\sum\limits_{n=1}^{\infty} \dfrac{n!}{n^n}$ by Ratio test

$$0 < \frac{c_{n+1}}{c_n} = \frac{(n+1)\,n!}{(n+1)\,(n+1)^n} \cdot \frac{n^n}{n!} = \left(\tfrac{n}{n+1}\right)^n \to e^{-1} < 1,$$

so the series is convergent and $\lim c_n = 0$.

Since $b_n = \dfrac{1}{c_n}$ and $b_n > 0, \lim b_n = +\infty$,so the sequence is divergent.

ALSO

from above $\dfrac{b_{n+1}}{b_n} = \left(\tfrac{n+1}{n}\right)^n > 1$, so the sequence is increasing,

there is NO upper bound and a lower bound is 0.

2. Determine whether the indicated series is absolutely convergent, conditionally convergent or divergent.

For a) $\sum\limits_{k=1}^{\infty} \dfrac{\arctan k}{1+k^2}$

Since $\frac{\pi}{4} = \arctan 1 \leq \arctan k < \frac{\pi}{2}$,so $0 < \dfrac{\arctan k}{1+k^2} < \dfrac{const}{k^2}$,

$\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ is convergent p-series , $p = 2 > 1$,

so By Comp.Test $\sum\limits_{k=1}^{\infty} \dfrac{\arctan k}{1+k^2}$ is convergent.

For b)

First abs.convergence $\sum_{n=1}^{\infty}(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}) = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + ... =$

$= 1 - \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 1 - 0 = 1$(It is a telescoping series.)

$\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$ is absolutely convergent .

For c)$\sum_{k=1}^{\infty} k^2 e^{-k}$ convergent by Ratio test

$$0 < \frac{a_{n+1}}{a_n} = \frac{(k+1)^2 e^{-k-1}}{k^2 e^{-k}} = \left(1 + \frac{1}{k}\right)^2 e^{-1} \to \frac{1}{e} < 1$$

For d) $\sum_{n=3}^{\infty} \frac{(-1)^n}{(n^2-2n)\sqrt{n}}$ is abs.convergent

since $\sum_{n=3}^{\infty} \frac{1}{(n^2-2n)\sqrt{n}} \sim \sum_{n=3}^{\infty} \frac{1}{n^{\frac{5}{2}}}$ which is a p-series where $p = \frac{5}{2} > 1$

to show the equivalence

$$\lim_{n\to\infty} \frac{\frac{1}{(n^2-2n)\sqrt{n}}}{\frac{1}{n^{\frac{5}{2}}}} = \lim_{n\to\infty} \frac{n^{\frac{5}{2}}}{n^{\frac{5}{2}} - 2n^{\frac{3}{2}}} \cdot \frac{n^{-\frac{5}{2}}}{n^{-\frac{5}{2}}} = \lim_{n\to\infty} \frac{1}{1-2n^{-1}} = 1 \neq 0 (\text{not } \infty)$$

3. Find the interval of convergence if

   (a) $\sum_{k=1}^{\infty} \frac{2^k}{\sqrt{k}} (x-1)^k$ b) $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$ (only abs.convergence)

   For a)

   $$\left|\frac{a_{k+1}}{a_k}\right| = \frac{2^{k+1}}{\sqrt{k+1}} \cdot \frac{\sqrt{k}}{2^k} = \sqrt{\frac{k}{k+1}} \cdot 2 \to 2, R = \frac{1}{2}$$

   and the series is abs.convergent on $\left(\frac{1}{2}, \frac{3}{2}\right)$

   For the ends $x = \frac{1}{2}$ or $\frac{3}{2}$ we get $\sum_{k=1}^{\infty} (-1)^n \frac{1}{\sqrt{k}}$ or $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

   the second one is.divergent since $p = \frac{1}{2} < 1$,but the first one is cond.convergent since the sequence $\frac{1}{\sqrt{k}} \searrow 0$

   Together the series is convergent on $\left[\frac{1}{2}, \frac{3}{2}\right)$

   For b)

   we investigate $\frac{c_{n+1}}{c_n} = \frac{(n+1)\,n!}{(n+1)\,(n+1)^n} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \left(1 - \frac{1}{n+1}\right)^n \to e^{-1}, R = e$

   and the series is abs.convergent on $(-e, e)$. The ends are difficult,Sterling formula!

4. **For a)** $\sum_{k=1}^{\infty} \frac{(\ell n\, 2)^k}{k!}$ we know that $e^s = \sum_{n=0}^{\infty} \frac{s^n}{n!}$ for any s.

So $e^s - 1 = \sum_{n=1}^{\infty} \frac{s^n}{n!}$ and for $s = \ln 2$ we get

$$\sum_{k=1}^{\infty} \frac{(\ell n\, 2)^k}{k!} = e^{\ln 2} - 1 = 2 - 1 = 1.$$

For b) $\sum_{n=3}^{\infty} \frac{(-1)^n}{2^n(n+1)}$ we need to know the sum of $\sum_{n=3}^{\infty} \frac{(x)^n}{(n+1)}$ for $x = -\frac{1}{2}$

and $\sum_{n=3}^{\infty} \frac{x^n}{(n+1)} = \frac{1}{x} \sum_{n=3}^{\infty} \frac{x^{n+1}}{(n+1)} = \frac{1}{x}\left[\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)} - \frac{x^3}{3} - \frac{x^2}{2} - \frac{x}{1}\right]$

and since we know that $\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)}$

$\sum_{n=3}^{\infty} \frac{x^n}{(n+1)} = -\frac{1}{x}\ln(1-x) - \frac{x^2}{3} - \frac{x}{2} - 1$ and finally for $x = -\frac{1}{2}$

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{2^n(n+1)} = 2\ln\tfrac{3}{2} - \tfrac{1}{12} + \tfrac{1}{4} - 1 = 2\ln\tfrac{3}{2} - \tfrac{1-3+12}{12} = 2\ln\tfrac{3}{2} - \frac{5}{6}.$$

5. (a) Determine if the indicated sequence is bounded, alternating or convergent

$a_n = \frac{2-(-1)^n}{n^2 - 2n}$ for $n \geq 3$

Since

$0 < \frac{1}{n(n-2)} \leq a_n \leq \frac{3}{n(n-2)} < 1$ positive terms,not alternating,bounded

and convergent to 0 by Squeeze Theorem.

(b) Is the sequence $c_n = \frac{3^n}{3^n - 2^n}$ convergent or monotonic?

$c_n = \frac{1}{1-(\frac{2}{3})^n} \to 1$ since $\left(\frac{2}{3}\right)^n \to 0$

also the geometric sequence$\left\{\left(\frac{2}{3}\right)^n\right\}$ is decreasing

so bottom of c_n is positive and increasing

finally c_n is positive and decreasing. Since the limit of c_n is NOT

zero the series $\sum_{n=1}^{\infty} c_n$ *is* divergent.

6. Find the Taylor series for $f(x) = \frac{1}{(x+3)x}$ around the center $x_0 = -1$,particularly the coefficient a_6.

For what values of x is the representation valid?

USE Partial fraction first

$$\frac{1}{(x+3)x} = \tfrac{1}{3}\left[\frac{-1}{x+3} + \frac{1}{x}\right] = \tfrac{1}{3}\left[\frac{-1}{(x+1)+2} + \frac{1}{(x+1)-1}\right] =$$

$$= \tfrac{1}{3}\left[\frac{-1}{2}\cdot\frac{1}{1+\frac{x+1}{2}} - \frac{1}{1-(x+1)}\right] =$$

(using $\dfrac{1}{1+r} = \displaystyle\sum_{n=0}^{\infty}(-r)^n$ for $r = \frac{x+1}{2}$ and $\dfrac{1}{1-r} = \displaystyle\sum_{n=0}^{\infty} r^n$ for $r = x+1$)

$$= -\tfrac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x+1}{2}\right)^n - \tfrac{1}{3}\sum_{n=0}^{\infty}(x+1)^n = \tfrac{-1}{3}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^{n+1}}+1\right](x+1)^n$$

for $-1 < \frac{x+1}{2} < 1$ $\quad -2 < x+1 < 2$ and $-1 < x+1 < 1$,together

$-2 < x < 0$. And $a_6 = -\frac{1}{3}\cdot\left[\frac{1}{2^7}+1\right] = -\frac{1+2^7}{3\cdot 2^7} = -\dfrac{129}{384}$.

7. Find Taylor polynomial of degree 3 for $f(x) = \ln\frac{x-1}{x}$ around the centre $x_0 = 2$.

We can find Taylor series first

$\ln\frac{x-1}{x} = \ln(x-1) - \ln x = \ln(x-2+1) - \ln(x-2+2) =$

$= \ln(1+(x-2)) - \ln 2\left(1+\frac{x-2}{2}\right) = \ln(1+(x-2)) - \ln 2 - \ln\left(1+\frac{x-2}{2}\right)$

(using $\ln(1+s) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n s^{n+1}}{(n+1)}$for $-1 < s \le 1$)

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^{n+1}}{(n+1)} - \ln 2 - \sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^{n+1}}{(n+1)2^{n+1}} =$$

$$= -\ln 2 + \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)}\left[1-\tfrac{1}{2^{n+1}}\right](x-2)^{n+1}$$

so $T_3(x) = -\ln 2 + \displaystyle\sum_{n=0}^{2}\frac{(-1)^n}{(n+1)}\left[1-\tfrac{1}{2^{n+1}}\right](x-2)^{n+1} =$

$= -\ln 2 + \frac{1}{2}(x-2) - \frac{3}{8}(x-2)^2 + \frac{7}{24}(x-2)^3$

OR

$a_0 = f(2) = \ln\frac{1}{2} = -\ln 2 \qquad a_1 = f'(2) = \frac{1}{2}$

since $f'(x) = [\ln(x-1) - \ln x]' = \frac{1}{x-1} - \frac{1}{x}$, then $f''(x) = \frac{-1}{(x-1)^2} + \frac{1}{x^2}$

and $a_2 = \frac{1}{2}f''(2) = -\frac{3}{8}$, finally $f'''(x) = \frac{2}{(x-1)^3} - \frac{2}{x^3}$

and $a_3 = \frac{1}{6}f'''(2) = \frac{1}{3}\cdot\left(1-\frac{1}{8}\right) = \frac{7}{24}$.

8. curve c given as the intersection of

the cone $\{\ z = \sqrt{2x^2+2y^2}\}$ and the plane $\{z + x = 1\}$.

from the plane $z = 1-x$ back to the cone $(1-x)^2 = 2x^2+2y^2$

$1 = x^2+2x+2y^2 \qquad 2 = (x+1)^2+2y^2 \qquad 1 = \left(\dfrac{x+1}{\sqrt{2}}\right)^2 + y^2$

and then a parametrization is $\qquad x = -1+\sqrt{2}\cos t, y = \sin t$ and

$z = 1 - x = 2 - \sqrt{2}\cos t \qquad t \in [0, 2\pi]$.

9. For the curve c given by $\mathbf{r}(t) = (2t, t^2, \ln t)\,, t > 0$ find

$\mathbf{r}'(t) = \left(2, 2t, \dfrac{1}{t}\right)$ and $t = 1$ for $P, t = e$ for R

then **for a)**

$\mathbf{d} = \mathbf{r}(1) = (2, 2, 1)$ and the tangent line is

$(x, y, z) = (2, 1, 0) + s\,(2, 2, 1)$ or $x = 2 + 2z$ and $y = 1 + 2z$

for b) for arclenght we need

$$\|\mathbf{r}'(t)\| = \sqrt{4 + 4t^2 + \frac{1}{t^2}} = \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} = \sqrt{\left(\frac{2t^2 + 1}{t}\right)^2} = \frac{2t^2 + 1}{|t|}$$

and $\displaystyle s = \int_1^e \frac{2t^2 + 1}{t} dt = \left[t^2 + \ln t\right]_1^e = e^2.$

10. For the curve c given by $\mathbf{r}\,(t) = (t\sin t, t\cos t, 2t)$

    (a) find an equation of the tangent line to c at the origin ;

    (b) find the arclength between the origin and the point $A\left(\frac{\pi}{2}, 0, \pi\right)$.

For a)

$\mathbf{r}'\,(t) = (\sin t + t\cos t, \cos t - t\sin t, 2)$ product rule

for the origin $t = 0$ so $\mathbf{d} = \mathbf{r}'(0) = (0, 1, 2)$

and an equation of the tangent is $(x, y, z) = t\,(0, 1, 2)$ or $x = 0, z = 2y$

For b)

for arclength we need $\|\mathbf{r}'\,(t)\|$

$\|\mathbf{r}'\,(t)\|^2 = (\sin t + t\cos t)^2 + (\cos t - t\sin t)^2 + 4 = \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t +$

$+ \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + 4 = 5 + t^2$

now , $t = 0$ for the origin and $t = \frac{\pi}{2}$ for the point A

so arclength $\displaystyle s = \int_0^{\frac{\pi}{2}} \|\mathbf{r}'\,(t)\|\, dt = \int_0^{\frac{\pi}{2}} \sqrt{5 + t^2} dt =$ (Table)=

$$= \left[\tfrac{t}{2}\sqrt{5 + t^2} + \tfrac{5}{2}\ln\left(t + \sqrt{5 + t^2}\right)\right]_0^{\frac{\pi}{2}} = \tfrac{\pi}{4}\sqrt{5 + \tfrac{\pi^2}{4}} + \tfrac{5}{2}\ln\left(\tfrac{\pi}{2} + \sqrt{5 + \tfrac{\pi^2}{4}}\right) - \tfrac{5}{2}\ln\sqrt{5}.$$

11. Find a parametrization of the curve c given as the intersection of two surfaces $c = \{x^2 + y^2 = 2z\,\} \cap \{3x - 4y - z = 0\}$.

from the plane $z = 3x - 4y$ into the paraboloid $x^2 + y^2 = 6x - 8y$

$x^2 - 6x + y^2 + 8y = (x - 3)^2 + (y + 4)^2 - 25 = 0$

so $\left(\frac{x-3}{5}\right)^2 + \left(\frac{y+4}{5}\right)^2 = 1$ thus a parametrization

$x = 3 + 5\cos t \qquad y = -4 + 5\sin t \qquad z = 25 + 15\cos t - 20\sin t, t \in [0, 2\pi)$.