FINAL HANDOUT
MATH 349 SOLUTION .

1. for $\displaystyle\sum_{n=1}^{\infty}\frac{(3x-1)^n}{3^{3n}\sqrt{n}}=\sum_{n=1}^{\infty}\frac{3^n(x-\frac{1}{3})^n}{3^{3n}\sqrt{n}}$.

   the centre is $c=\frac{1}{3}$ and the coefficients $a_n=\dfrac{1}{3^{2n}\sqrt{n}}$

   the radius of convergence $R=\dfrac{1}{L}=9$

   since $L=\displaystyle\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{3^{2n}\sqrt{n}}{3^{2n+2}\sqrt{n+1}}=\frac{1}{9}\lim_{n\to\infty}\sqrt{\frac{n}{n+1}}=\frac{1}{9}$

   thus the series is absolutely convergent on $\left(\frac{1}{3}-9,\frac{1}{3}+9\right)=\left(\frac{-26}{3},\frac{28}{3}\right)$

   ends: $x=\frac{1}{3}+9$ gives $\displaystyle\sum_{n=1}^{\infty}\frac{\left(x-\frac{1}{3}\right)^n}{3^{2n}\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which is divergent p-series $p=\frac{1}{2}<1$

   for $x=\frac{1}{3}-9$ gives $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ which is cond.convergent since the sequence $\frac{1}{\sqrt{n}}\searrow 0$

   (positive,decr.since $\frac{1}{\text{incr.}}$, and limit is $\frac{1}{\infty}=0$)

   together the interval of convergence is $\left[\frac{-26}{3},\frac{28}{3}\right)$, outside the series is divergent.

2. we know that $\displaystyle e^s=\sum_{n=0}^{\infty}\frac{s^n}{n!}$ for any s ,use $s=x+1$

   then $\displaystyle f(x)=xe^x=[(x+1)-1]\,e^{x+1}e^{-1}=\frac{1}{e}(x+1)\sum_{n=0}^{\infty}\frac{(x+1)^n}{n!}-\frac{1}{e}\sum_{n=0}^{\infty}\frac{(x+1)^n}{n!}=$

   $\displaystyle=\frac{1}{e}\sum_{n=0}^{\infty}\frac{(x+1)^{n+1}}{n!}-\frac{1}{e}\sum_{n=0}^{\infty}\frac{(x+1)^n}{n!}=\frac{1}{e}\sum_{k=1}^{\infty}\frac{(x+1)^k}{(k-1)!}-\frac{1}{e}\sum_{n=0}^{\infty}\frac{(x+1)^n}{n!}=$

   $\displaystyle=-\frac{1}{e}+\frac{1}{e}\sum_{k=1}^{\infty}\left[\frac{1}{(k-1)!}-\frac{1}{k!}\right](x+1)^k=-\frac{1}{e}+\frac{1}{e}\sum_{k=1}^{\infty}\frac{k-1}{k!}(x+1)^k$ for any x.

3. For the curve $c=\{z=x^2+y^2\}\cap\{6x-2y-z=1\}$

   for a) $z=x^2+y^2=6x-2y-1$ so $x^2+y^2-6x+2y+1=0$,complete the squares

   $(x-3)^2+(y+1)^2=9$ and $\left(\frac{x-3}{3}\right)^2+\left(\frac{y+1}{3}\right)^2=1$ so

   $x=3+3\cos t, y=-1+3\sin t, z=6x-2y-1=19+18\cos t-6\sin t$, $t\in[0,2\pi)$

   for b)we have two options to find $\overrightarrow{d}$ in $(x,y,z)=(0,-1,1)+t\,\overrightarrow{d}$

   from part a:

   $t=\pi$ for the point $P=\overrightarrow{r}(\pi)$ and $\overrightarrow{d}=\overrightarrow{r}'(\pi)=(0,-3,6)$ or $(0,1,-2)$

   since $\overrightarrow{r}'(t)=(-3\sin t,3\cos t,-18\sin t-6\cos t)$

   OR

$\overrightarrow{d} = \overrightarrow{n_1} \times \overrightarrow{n_2}$ where $\overrightarrow{n_1} = \nabla F = (2x, 2y, -1)$ then at P $\overrightarrow{n_1} = (0, -2, -1)$
where $F(x,y,z) = x^2 + y^2 - z = 0$ and $G(x,y,z) = 6x - 2y - z = 1$
and $\overrightarrow{n_2} = \nabla G = (6, -2, -1)$ so $(x,y,z) = (0,-1,1) + t(0,1,-2)$.

for c) for P $\quad t = \pi$ and for R $\quad t = \dfrac{\pi}{2}$,also

$$\|\overrightarrow{r}'(t)\| = \sqrt{9 + 36(3\sin t + \cos t)^2} = \sqrt{45 + 36(8\sin^2 t + 6\sin t\cos t)}$$

so $\quad s = \displaystyle\int_{\frac{\pi}{2}}^{\pi} \sqrt{45 + 72(4\sin^2 t + 3\sin t\cos t)}dt.$

4. the equation $z = f(1,-1) + \nabla f(1,-1) \bullet (x-1, y+1) = 1 - (x-1)$ gives
   $x + z = 2$ $\quad$ since for $f(x,y) = e^{yx^2 \ln x}$ $\quad f(1,-1) = 1,$
   $f_x(x,y) = e^{yx^2 \ln x}(2xy\ln x + yx)$ $\quad f_x(1,-1) = -1$
   $f_y(x,y) = e^{yx^2 \ln x} x^2 \ln x$ $\quad f_y(1,-1) = 0$
   OR $\overrightarrow{n} = (\nabla f(1,-1), -1) = (-1, 0-1)$ and the point is $(1,-1,f(1,-1)) = (1,-1,1)$
   so $-x - z = d$ and through $P(1,-1,1)$ $\quad x + z = 2.$

5. **for a)** at the origin we have to use the definition:
   $$f_x(0,0) = \lim_{x\to 0}\frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0}\frac{1-0}{x} = \lim_{x\to 0^{\pm}}\frac{1}{x} \qquad DNE\ (\pm\infty)$$
   $$f_y(0,0) = \lim_{y\to 0}\frac{f(0,y) - f(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$
   thus the gradient does not exist at (0,0).
   for b) since $f(x,x) = \frac{1}{4}$ for any $x \neq 0$ $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) \neq 0 = f(0,0)$
   the function is discont. at $(0,0)$.

6. for the domain $\dfrac{x}{y} > 0$ both positive OR both negative
   so the domain consists of the first and third quadrants without the axes
   for level curves for any c
   $c = \ln\frac{x}{y}, e^c = \frac{x}{y}, y = e^{-c}x$ $\quad$ lines through the origin
   without the origin and with positive slopes $m = e^{-c} = 1, e, \frac{1}{e}$
   since we got a level curve for any c the range is $(-\infty, \infty)$.

7. $\nabla f = (2xe^{-y}, -e^{-y}(x^2 + \cos z), -e^{-y}\sin z)$ all partials are cont.functions so
   $D_v f = \nabla f \bullet \boldsymbol{v}$ where $\boldsymbol{v}$ is the unit vector in the direction of $\overrightarrow{AB} = (-3,-1,-\pi)$
   and $\left\|\overrightarrow{AB}\right\| = \sqrt{10 + \pi^2}, \nabla f(A) = (4,-3,0)$ and finally
   $$D_v f(A) = \tfrac{1}{\sqrt{10+\pi^2}}(4,-3,0) \bullet (-3,-1,-\pi) = \frac{-9}{\sqrt{10+\pi^2}}.$$

8. $\dfrac{\partial F}{\partial u}(u,v) = \nabla f(x,y) \bullet \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right) = (f_x, f_y) \bullet \left(v^2, \frac{1}{v}\right)$, where $x = uv^2, y = \frac{u}{v}$.
now for $u = 2, v = -1$ $\quad x = 2, y = -2$ so we need $f_x(2,-2) = 3, f_y(2,-2) = -2$
and $\dfrac{\partial F}{\partial u}(2,-1) = (3,-2) \bullet (1,-1) = 5.$

9. **for a)**
for $T = 0$ $\quad y^2 - 3x^2 = 0$ $\quad y = \pm x\sqrt{3}$ $\quad$ two lines
for $T = 3$ $\quad y^2 - 3x^2 = 3$ $\quad$ hyperbola with intercepts $y = \pm\sqrt{3}, x = 0$
for $T = 6$ $\quad y^2 - 3x^2 = -6$ $\quad$ hyperbola with intercepts $x = \pm\sqrt{2}, y = 0$
for b)
the direction is $-\nabla T(-1,2)$
so first $\nabla T = (-6x, 2y)$ then $\nabla T(-1,2) = (6,4) = 2(3,2)$ so $\boldsymbol{v} = \frac{-1}{\sqrt{13}}(3,2)$.

10. Define $F_1(x,y,z) = xy^2 - z + u^2, F_2(x,y,z) = x^3z + 2y - u$
$F_3(x,y,z) = xu + y - xyz$.
since **all partials are continuous function** the only condition is that

$$\left\|\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)}\right\| = \det\begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{bmatrix} \neq 0 \text{ at the given point}$$

so $\det\begin{bmatrix} y^2 & 2xy & -1 \\ 3x^2z & 2 & x^3 \\ u - yz & 1 - xz & -xy \end{bmatrix}$ at the given point $= \det\begin{bmatrix} 1 & 2 & -1 \\ -3 & 2 & 1 \\ 2 & 2 & -1 \end{bmatrix} =$

$= 1 \cdot \det\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} - (-3) \cdot \det\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} + 2 \cdot \det\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = -4 + 0 + 8 = 4 \neq 0$

11. For $\quad f(x,y) = \ln(y\cos x)$ $x = 0, y = 1$
$T_2(x,y) = f(0,1) + \nabla f(0,1) \bullet (x, y-1) + \frac{1}{2}[Ax^2 + 2Bx(y-1) + C(y-1)^2]$
so $f(0,1) = \ln 1 = 0$, since $f(x,y) = \ln y + \ln\cos x$ $\quad f_x = -\frac{\sin x}{\cos x} = -\tan x$
$f_y = \dfrac{1}{y}$ $\quad f_{xx} = -\sec^2 x = \dfrac{-1}{\cos^2 x}$ $\quad f_{xy} = 0$ $\quad f_{yy} = -\dfrac{1}{y^2}$
and $\quad A = f_{xx}(0,1) = -1, B = f_{xy} = 0$ $\quad C = f_{yy}(0,1) = -1$
thus
$T_2(x,y) = 0 + (0,1) \bullet (x, y-1) + \frac{1}{2}[-x^2 + 0x(y-1) - (y-1)^2] =$
$= y - 1 - \dfrac{x^2}{2} - \dfrac{(y-1)^2}{2}.$

12. Define $F(x,y,z) = \arcsin(zy) + z^3x + x^2y + 8$

then $\dfrac{\partial F}{\partial x} = z^3 + 2xy \qquad \dfrac{\partial F}{\partial y} = \dfrac{z}{\sqrt{1-z^2y^2}} + x^2$ and $\dfrac{\partial F}{\partial z} = \dfrac{y}{\sqrt{1-z^2y^2}} + 3z^2x$

all functions are cont. at the point $P(-1,0,2)$ and $\dfrac{\partial F}{\partial z}(P) = -12 \neq 0$ so the equation can be solved for z as a function of x,y around he point $P(-1,0,2)$

then $\dfrac{\partial z}{\partial x} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\dfrac{2xy+z^3}{\frac{y}{\sqrt{1-z^2y^2}} + 3z^2x} = \dfrac{-8}{-12} = \dfrac{2}{3}$ at P and

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{\frac{z}{\sqrt{1-z^2y^2}} + x^2}{\frac{y}{\sqrt{1-z^2y^2}} + 3z^2x} = -\frac{3}{-12} = \frac{1}{4}$$ at that point..