DEPARTMENT OF MATHEMATICS AND STATISTICS

MIDTERM EXAM

MATH 349 LEC 02

Fall 2006 **TIME: 90 minutes**

Name:______________________________ **I.D. No.**:____________

Total 60 **Each question is for 10 points.**

1. Is the sequence $a_n = \dfrac{\sqrt{n}}{n^2+12}$ convergent,ultimatelly monotonic,alternating,bounded? Find the limit and an uper and a lower bound.

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n^2+12} = (\text{ L'H.R.}) = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{2x} = \lim_{x\to\infty} \frac{1}{4x\sqrt{x}} = 0$$

$a_n > 0$ so the sequence is NOT alternating ,it is **convergent** thus **bounded**

for monotonicity

$$f'(x) = \left(\frac{\sqrt{x}}{x^2+12}\right)' = \frac{\frac{1}{2\sqrt{x}}(x^2+12) - \sqrt{x}2x}{(x^2+12)^2} = \frac{(x^2+12)-4x^2}{2\sqrt{x}(x^2+12)^2} = \frac{12-3x^2}{2\sqrt{x}(x^2+12)^2}$$

if $x > 2$ the top is negative , bottom positive so $f' < 0$ and the sequence is **ult.decr.**

therefore $\quad 0 < a_n \le a_2 < 1$

2. Is the series $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n \ln^2 n}{n}$ absolutely or conditionally convergent or divergent? Explain.

first

$$\sum_{n=3}^{\infty} \left|\frac{(-1)^n \ln^2 n}{n}\right| = \sum_{n=3}^{\infty} \frac{(\ln n)^2}{n} = \infty \text{ by Integral test}$$

$$\int_3^{\infty} \frac{(\ln x)^2}{x} dx = \left[\frac{1}{3}(\ln x)^3\right]_3^{\infty} = \infty \text{ (subst. } u = \ln x)$$

OR by Comparison test $\dfrac{\ln^2 n}{n} \ge \dfrac{\ln^2 3}{n}$ and series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n} = \infty$

now, try Alt.Test $\quad a_n = \dfrac{\ln^2 n}{n}$ is ult.decr. sequence

since: $f'(x) = \left(\dfrac{\ln^2 x}{x}\right)' = \dfrac{2\ln x - \ln^2 x}{x^2} = \dfrac{\ln x(2-\ln x)}{x^2} < 0$ if $2 < lnx$

for $x > e^2$ thus for $n \ge 8$

and $\displaystyle\lim_{n\to\infty} a_n = (\text{by L'H.R.twice}) \lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{2}{x} = 0$ $\quad$ the series is **cond.convergent.**

3. Find the Taylor series around $c = 1$ for the function $f(x) = x + \ln x$.Where is the expansion valid?

   Then find the values of a_1 and a_4.

   we are looking for a_n such that $f(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$

   first $f(x) = 1 + (x-1) + \ln(1 + (x-1)) =$

   (using $\ln(1+s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} s^n$ for $-1 < s \leq 1$)

   $= 1 + (x-1) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n = 1 + 2(x-1) + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$

   for $-1 < x - 1 \leq 1$ $0 < x \leq 2$ $a_1 = 2, a_4 = -\frac{1}{4}$.

   check: $a_0 = f(1) = 1$ $a_1 = f'(1) = 2$ since $f'(x) = 1 + \frac{1}{x}$

   also $f''(x) = -\frac{1}{x^2}, f'''(x) = \frac{2}{x^3}, f^{(4)}(x) = -\frac{6}{x^4}$and $a_4 = \frac{f^{(4)}(1)}{4!} = -\frac{1}{4}$.

4. Find the sum of (a) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1} 9^n}{(2n)!}$; (b) $\sum_{n=4}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$.

   for a) recall $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for any x

   $\sum_{n=2}^{\infty} \frac{(-1)^{n+1} 9^n}{(2n)!} = -\sum_{n=2}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!} = 1 - \frac{9}{2} - \cos 3 = -\frac{7}{2} - \cos 3.$

   for b) telescoping

   $\sum_{n=4}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) = \frac{1}{2} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + .. \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{2}.$

5. For the curve c given by $\mathbf{r}(t) = (e^t \cos t, e^t \sin t)$

   find the arclength between the points $A(1,0)$ and $B\left(0, e^{\frac{\pi}{2}}\right)$.

   first $t = 0$ for A and $t = \frac{\pi}{2}$ for B

   then $\mathbf{r}'(t) = (e^t \cos t - e^t \sin t, e^t \cos t + e^t \sin t) = e^t (\cos t - \sin t, \cos t + \sin t)$

   and

   $\|\mathbf{r}'(t)\| = e^t \sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2} = e^t \sqrt{2}$,also $(e^t)^2 = e^{2t}, \sqrt{e^{2t}} = e^t$

   finally, the arclength $s = \int_0^{\frac{\pi}{2}} \|\mathbf{r}'(t)\| \, dt = \sqrt{2}\left(e^{\frac{\pi}{2}} - 1\right)$.

6. Find a parametrization of the curve c given as the intersection of two surfaces

   $c = \{x^2 + y^2 = z\} \cap \{2x - 4y + z = 4\}$.

   Then find an equation of the tangent line to c at $P(2,2,8)$

from the plane $z = 4 - 2x + 4y$ into the paraboloid $x^2 + y^2 = 4 - 2x + 4y$

$x^2 + 2x + y^2 - 4y = 4 \qquad (x+1)^2 + (y-2)^2 = 9$ and $\left(\frac{x+1}{3}\right)^2 + \left(\frac{y-2}{3}\right)^2 = 1$

so $\dfrac{x+1}{3} = \cos t \qquad \dfrac{y-2}{3} = \sin t$

$x = -1 + 3\cos t \qquad y = 2 + 3\sin t \qquad z = 14 - 6\cos t + 12\sin t \qquad t \in [0, 2\pi]$.

$\mathbf{r}(t) = (-1 + 3\cos t, 2 + 3\sin t, 14 - 6\cos t + 12\sin t)$

$\mathbf{r}'(t) = (-3\sin t, 3\cos t, 6\sin t + 12\cos t)$

for $P(2, 2, 8) \qquad t = 0 \qquad$ then $\mathbf{d} = \mathbf{r}'(0) = (0, 3, 12)$or $(0, 1, 4)$

and

$(x, y, z) = (2, 2, 8) + t(0, 1, 4)$