THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL EXAMINATION
MATH 349 (L02)
Fall, 2006

Time: 3 hours

NOTE: A calculator *IS* allowed.Each question is for 10 marks.

1. $\sum_{n=2}^{\infty} \frac{\sqrt{n}\,(3x-1)^n}{\ln n} = \sum_{n=2}^{\infty} \frac{\sqrt{n}3^n\left(x-\frac{1}{3}\right)^n}{\ln n}$ so the center is $c=\frac{1}{3}, a_n = \frac{\sqrt{n}3^n}{\ln n}$ and $R=\frac{1}{L}$

   where $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}}.\frac{\ln n}{\ln(n+1)} \cdot 3 = 3$ and $R=\frac{1}{3}$

   thus the series is **abs.convergent** on $\left(0,\frac{2}{3}\right)$ and **divergent** on $(\infty,0)\cup\left(\frac{2}{3},\infty\right)$,

   could be conditionally convergent at the ends

   at $x=0$ $\quad \sum_{n=2}^{\infty} \frac{\sqrt{n}\,(3x-1)^n}{\ln n} = \sum_{n=2}^{\infty} \frac{\sqrt{n}\,(-1)^n}{\ln n}$ and at $x=\frac{2}{3}$ $\quad \sum_{n=2}^{\infty} \frac{\sqrt{n}\,(3x-1)^n}{\ln n} =$

   $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$

   both are **divergent** since $\lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{2} = \infty \neq 0$

2. Looking for or $f(x) = \frac{x}{x+2} = \sum_{n=0}^{\infty} a_n\,(x-1)^n \qquad \frac{x}{x+2} = \frac{x+2-2}{x+2} = 1 - \frac{2}{x-1+3}$

   and finally $f(x) = 1 - \frac{2}{3}\cdot\frac{1}{1+\frac{x-1}{3}} = 1 - \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-1}{3}\right)^n = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{2\,(-1)^{n+1}}{3^{n+1}}\,(x-1)^n$

   thus $a_0 = f(1) = \frac{1}{3}$ and $a_3 = \frac{2}{3^4} = \frac{2}{81}$

   the represenatation is valid for $-1 < \frac{x-1}{3} < 1 \qquad -2 < x < 4.$

3. c given by $\mathbf{r}(t) = (\sin(\pi t), \cos(\pi t), t^2)$ for $A\,(0,1,0)$ $t=0$ and for $B\,(0,-1,1)$ $t=1$

   $\mathbf{r}'(t) = (\pi\cos(\pi t), -\pi\sin(\pi t), 2t) \qquad \|\mathbf{r}'(t)\| = \sqrt{\pi^2+4t^2}$

   and $\quad s = \int_0^1 \|\mathbf{r}'(t)\|\,dt = \int_0^1 \sqrt{\pi^2+4t^2}dt$ (by subst.$u=2t$) $= \frac{1}{2}\int_0^2 \sqrt{\pi^2+u^2}du =$

   (Table)$=\frac{1}{4}\left[u\sqrt{\pi^2+u^2} + \pi^2\ln(u+\sqrt{\pi^2+u^2}\right]_0^2 = \frac{1}{4}\left[u\sqrt{\pi^2+4} + \pi^2\ln\frac{1}{\pi}(4+\sqrt{\pi^2+4})\right].$

4. Given that $f(x, y)$ is differentiable everywhere and

   $f_x(-1,2) = 11, \quad f_x(5,3) = -7, \quad f_y(-1,2) = 5, \quad f_y(5,3) = 9$

   find $\frac{\partial h}{\partial u}, \frac{\partial h}{\partial v}$ at the point $(-1,2)$ if $h(u,v) = f(x,y)$ where $x = v^2 - u, y = u^2 + v$.

   for $u = -1, v = 2$ $\quad x = 5, y = 3$;also $\frac{\partial x}{\partial u} = -1, \frac{\partial x}{\partial v} = 2v, \frac{\partial y}{\partial u} = 2u, \frac{\partial y}{\partial v} = 1$

   we can use Chain Rule

   $\frac{\partial h}{\partial u}(-1,2) = f_x(5,3)\frac{\partial x}{\partial u}(-1,2) + f_y(5,3)\frac{\partial y}{\partial u}(-1,2) = (-7)(-1) + 9(-2) = -11;$

   $\frac{\partial h}{\partial v} = f_x(5,3)\frac{\partial x}{\partial v}(-1,2) + f_y(5,3)\frac{\partial y}{\partial v}(-1,2) = (-7)(4) + 9(1) = -19.$

5. The temperature on a metal plate is given by $T(x,y) = 4e^{-\left(2x^2+y^2\right)}$ at the point (x,y).

   (a) Find the isobars (level curves) for $T = 4, \frac{1}{4}$;

   (b) Find the directions of NO change in the heat on the plate from the point (3.5);

   (c) Find the direction of the greatest decrease in the heat on the plate from (3,5).

   for a)

   generally $\quad c = 4e^{-\left(2x^2+y^2\right)} \quad \ln\frac{c}{4} = -(2x^2 + y^2) \quad 2x^2 + y^2 = \ln\frac{4}{c}$

   we will get ellipses if $\ln\frac{4}{c} > 0 \quad 4 > c > 0$

   so if $c = 4 \quad x = 0, y = 0$ only origin

   if $c = 8$ nothing; if $c = \frac{1}{4} \quad 2x^2 + y^2 = \ln 16$

   for b)

   $T_x = 4e^{-\left(2x^2+y^2\right)}(-4x) \quad T_y = 4e^{-\left(2x^2+y^2\right)}(-2y)$ at $(3,5)$

   $\nabla T(3,5) = 8e^{-43}(-6,-5)$

   looking for $\mathbf{u}$ such that $D_{\mathbf{u}}f(3,5) = \nabla T(3,5) \cdot \mathbf{u} = 0$ i.e.$\mathbf{u}$ is perpendicular to $\nabla T(3,5)$

   $\mathbf{u}$ is perpendicular to $(-6,-5)$ thus $\mathbf{u} = \frac{1}{\sqrt{61}}(5,-6)$ or $\frac{1}{\sqrt{61}}(-5,6)$

   for c)

   looking for $\mathbf{u}$ such that $D_{\mathbf{u}}f(3,5) = \nabla f(3,5) \cdot \mathbf{u}$ is the biggest negative (smallest)

   $\mathbf{u}$ is parallel to $-\nabla f(3,5) \quad \mathbf{u} = \frac{1}{\sqrt{61}}(6,5)$.

6. Show that the system of equations $x = 3u - v^3 \quad y = u^3 + 3v$ could be always solved for u and v in terms of x, y.

   Then find $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ at that point $u = 0, v = -1$.

   Re-write the system $F_1 = 3u - v^3 - x = 0 \quad F_2 = u^3 + 3v - y = 0$

   all partials $\frac{\partial F_1}{\partial u} = 3 \quad \frac{\partial F_1}{\partial v} = -3v^2 \quad \frac{\partial F_1}{\partial x} = -1 \quad \frac{\partial F_1}{\partial y} = 0;$

$$\frac{\partial F_2}{\partial u} = 3u^2 \qquad \frac{\partial F_2}{\partial v} = 3 \qquad \frac{\partial F_1}{\partial x} = 0; \frac{\partial F_1}{\partial y} = -1 \text{ are cont.}$$

and $\frac{\partial (F_1, F_2)}{\partial (u, v)} = \det \begin{bmatrix} 3 & -3v^2 \\ 3u^2 & 3 \end{bmatrix} = 9 + 9u^2v^2 > 0$ always

so By Impl.Funct.Theorem the systen is solvable for u and v;

now $$\frac{\partial u}{\partial x} = -\frac{\frac{\partial (F_1, F_2)}{\partial (x, v)}}{\frac{\partial (F_1, F_2)}{\partial (u, v)}} = \frac{-1}{9 + 9u^2v^2} \det \begin{bmatrix} -1 & -3v^2 \\ 0 & 3 \end{bmatrix} = \frac{3}{9 + 9u^2v^2} = \frac{1}{3}$$

at the given point; similarily

$$\frac{\partial v}{\partial y} = -\frac{\frac{\partial (F_1, F_2)}{\partial (u, y)}}{\frac{\partial (F_1, F_2)}{\partial (u, v)}} = \frac{-1}{9 + 9u^2v^2} \det \begin{bmatrix} 3 & 0 \\ 3u^2 & -1 \end{bmatrix} = \frac{3}{9 + 9u^2v^2} = \frac{1}{3}.$$

7. Find an equation of the tangent plane at $P(1, -1, e)$ to the surface $x^y - z^x = 1 - e$.

.for $F(x, y, z) = x^y - z^x$ find ∇F

$F_x = yx^{y-1} - z^x \ln z \qquad F_y = x^y \ln x \qquad F_z = -xz^{x-1}$ at the poin P

$\nabla F(P) = (-1 - e, 0, -1) = \mathbf{n} \qquad -(1 + e)(x - 1) + 0 - (z - e) = 0$

or $\quad (1 + e) x + z = 1 - 2e.$

8. Find the Taylor polynomial $T_2(x, y)$ for the function $f(x, y) = \ln(x - \sin y)$around the center $(1, 0)$ then use it to estimate $f\left(\frac{3}{4}.\frac{1}{4}\right)$.

we need $f(1, 0) = \ln 1 = 0$ then

$$f_x = \frac{1}{x - \sin y} \qquad f_y = \frac{-\cos y}{x - \sin y} \qquad f_x(1, 0) = 1 \qquad f_y(1, 0) = -1 \text{ also}$$

$$f_{xx} = \frac{-1}{(x - \sin y)^2} \qquad f_{xy} = f_{yx} = \frac{\cos y}{(x - \sin y)^2} \qquad f_{yy} = \frac{\sin y}{x - \sin y} - \frac{\cos^2 y}{(x - \sin y)^2}$$

$A = f_{xx}(1, 0) = -1 \qquad B = f_{xy} = f_{yx}(1, 0) = 1 \qquad C = f_{yy}(1, 0) = -1$

then $T_2(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0) y + \frac{1}{2}\left[A(x - 1)^2 + 2B(x - 1) y + Cy^2\right]$

$T_2(x, y) = (x - 1) - y + \frac{1}{2}\left[-(x - 1)^2 + 2(x - 1) y - y^2\right]$ and $f(x, y) \doteq T_2(x, y)$ around $(1, 0)$

$$f\left(\tfrac{3}{4}.\tfrac{1}{4}\right) \doteq T_2(\tfrac{3}{4}, \tfrac{1}{4}) = \tfrac{-1}{4} - \tfrac{1}{4} - \tfrac{1}{32} - \tfrac{1}{16} - \tfrac{1}{32} = -\frac{5}{8} = -0.625$$

9. For the function $f(x, y) = \dfrac{x^2 y}{x^2 + y^2}$

   (a) find the gradient ∇f at the point $A(-1, 1)$;

   (b) find the directional derivative $D_{\mathbf{u}} f$ at the point $A(-1, 1)$ in the direction from A to $B(-2, 5)$.

partials

$$f_x = \frac{2xy}{x^2+y^2} - \frac{2x^3y}{(x^2+y^2)^2} \qquad f_y = \frac{x^2}{x^2+y^2} - \frac{2x^2y^2}{(x^2+y^2)^2} \text{ .at } A$$

$\nabla f\ (-1,1) = \left(\frac{-1}{2}, 0\right)$ since partilas are cont $D_{\mathbf{u}}f(-1,1) = \nabla f(-1,1)\cdot\mathbf{u}$

where $\overrightarrow{AB} = (-1,4)$ then $\mathbf{u} = \frac{1}{\sqrt{17}}(-1,4)$ and $D_{\mathbf{u}}f(-1,1) = (\frac{-1}{2}, 0)\cdot\mathbf{u} = \dfrac{1}{2\sqrt{17}}$.

10. The function f defined as follows

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{at } (0,0) \end{cases}$$

   (a) Is f continuous at$(0,0)$?
   (b) Find ∇f at $(0,0)$;
   (c) Find the directional derivative $D_{\mathbf{u}}f$ at $(0,0)$ where $\mathbf{u} = \frac{1}{\sqrt{2}}(1,-1)$ if it exists.
   (d) Is f differentiable at $(0,0)$? Explain.

for a)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0 \text{ since } \left|\frac{x^2y}{x^2+y^2}\right| \le \left|\frac{x^2y}{x^2}\right| \le |y| \text{ for } x \neq 0 \text{ and } f(0,y) = 0$$

so f is cont. at $(0,0)$;

for b) and c) we cannot use the rules as in question 9,we have to use Definitions

$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0 \text{ also}$$

$$D_{\mathbf{u}}f(0,0) = \tfrac{1}{\sqrt{2}}\lim_{t\to 0}\frac{f(t,-t)-f(0,0)}{t} = \frac{1}{\sqrt{2}}\lim_{t\to 0}\frac{\frac{-t^3}{2t^2}-0}{t} = -\frac{1}{2\sqrt{2}}$$

since $D_{\mathbf{u}}f(0,0) \neq \nabla f(0,0)\cdot\mathbf{u}$ =0 the function is NOT diif. at $(0,0)$.

End of Examination