

MATH 349
Handout # 1-solution

A

FOR 1)

For $a_n = \frac{e^n}{1+3^n}$ we can estimate: $0 < a_n < \frac{e^n}{3^n} = \left(\frac{e}{3}\right)^n$

by a geometric sequence with $r = \frac{e}{3} < 1$ so the limit is 0 and by Squeeze theorem also $\lim a_n = 0$.

Back to the estimate : $0 < a_n < 1$ so a lower bound is 0 and an upper bound is 1 we can use L'Hopital Rule for the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{e^x}{1+3^x} = \lim_{x \rightarrow \infty} \frac{e^x}{3^x \cdot \ln 3} = \frac{1}{\ln 3} \lim_{x \rightarrow \infty} e^{x(1-\ln 3)} = "e^{-\infty}" = 0 \text{ since } \ln 3 > 1.$$

FOR 2)

Let's call the sequence $x_n = \frac{2^n}{n!}$

then $0 < \frac{x_{n+1}}{x_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} < 1$ thus

$0 < x_{n+1} < x_n$ for all $n > 1$ so $0 < x_n < x_1 = 2$

therefore the sequence is bounded and decreasing. Thus the limit exists.

To find it estimate $x_n = \frac{2 \cdot 2 \cdot \dots \cdot 2 \cdot 2}{n(n-1) \dots \cdot 2 \cdot 1} \leq \frac{2}{n} \cdot 1 \dots 2 = \frac{4}{n}$ for $n > 2$

so by Squ.Th. the limit is 0.

FOR 3)

for a) $\{1, 2, 1, 3, 1, 4, 1, \dots, 1, n, 1, \dots\}$ or $\{n + (-1)^n\}_{n=1}^{\infty}$ or $\{(-1)^n n\}_{n=1}^{\infty}$

for b) $\{(-1)^n\}$ or $\{1, 2, 1, 2, 1, \dots\}$

B

FOR 1)

For $a_n = \frac{n!}{n^n}$ we can estimate:

$$0 < \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n+1}{n+1} \left(\frac{n}{n+1}\right)^n < 1$$

thus $a_{n+1} < a_n$ so the sequence is decreasing

and $0 < a_n < a_1 = 1$ for all $n > 1$ so the sequence is bounded

for the limit estimate $0 < a_n = \frac{n!}{n^n} = \frac{n(n-1) \dots 2 \cdot 1}{n \cdot n \cdot \dots n \cdot n} \leq \frac{1}{n}$

so the limit is 0 by Squ.Th.

FOR 2)

For $a_n = \sqrt{n^2 - \frac{n}{3}} - n$ the type of the limit is " $\infty - \infty$ " so rationalize to change it to the type " $\frac{\infty}{\infty}$ ",

then divide both the top and bottom by the highest power in the denominator:

$$\left(\sqrt{n^2 - \frac{n}{3}} - n\right) \cdot \frac{\sqrt{n^2 - \frac{n}{3}} + n}{\sqrt{n^2 - \frac{n}{3}} + n} = \frac{n^2 - \frac{n}{3} - n^2}{\sqrt{n^2 - \frac{n}{3}} + n} = \frac{-\frac{n}{3}}{\sqrt{n^2 - \frac{n}{3}} + n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} =$$

$$\left(\text{using } \frac{1}{n} = \frac{1}{\sqrt{n^2}}\right) = \frac{-\frac{1}{3}}{\sqrt{1 - \frac{1}{3n}} + 1} = \frac{-\frac{1}{3}}{2} \rightarrow -\frac{1}{6}.$$

Since the sequence is convergent it must be bounded. From the last simplification

$$a_n = \frac{-\frac{1}{3}}{\sqrt{1 - \frac{1}{3n}} + 1} \text{ so } a_n < 0 \quad \text{upper bound, and } \sqrt{1 - \frac{1}{3n}} + 1 \geq 1$$

$$(\text{ since } \sqrt{\dots} \text{ is always } \geq 0) \quad \frac{1}{\sqrt{1 - \frac{1}{3n}} + 1} \leq 1$$

$$\text{thus } a_n \geq -\frac{1}{3} \quad \text{lower bound. Together} \quad -\frac{1}{3} \leq a_n < 0.$$

FOR 3) for a) $\left\{-\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{-1, -\frac{1}{2}, -\frac{1}{3}, \dots\right\}$
for b) $\{(-1)^n n\}_{n=1}^{\infty} = \{-1, 2, -3, 4, -5, \dots\}$

C

FOR 1).

$$\text{For } a_n = \frac{n + (-1)^n}{n} \quad a_1 = 0, a_2 = \frac{3}{2}, a_3 = \frac{2}{3}, a_4 = \frac{5}{4} \dots,$$

also $a_n = 1 + \frac{(-1)^n}{n}$, all terms positive except the first one, so not alternating, but convergent since $\left|\frac{(-1)^n}{n}\right| \leq \frac{1}{n}$ and the limit of $\frac{1}{n}$ is 0.

So $\lim_{n \rightarrow \infty} a_n = 1$. Thus the sequence must be bounded $0 \leq a_n \leq 1 + \frac{1}{n} \leq \frac{3}{2}$.

Now, for even n : $a_n = 1 + \frac{(-1)^n}{n} = 1 + \frac{1}{n} > 1$

and for odd n : $a_n = 1 + \frac{(-1)^n}{n} = 1 - \frac{1}{n} < 1$ so the sequence is not monotonic.

FOR 2)

to find the limit of $a_n = \frac{4^n}{2^n + 10}$ divide by 2^n the top and bottom

$$a_n = \frac{\frac{4^n}{2^n}}{1 + \frac{10}{2^n}} = \frac{2^n}{1 + \frac{10}{2^n}} \rightarrow \frac{\infty}{1 + 0} = +\infty$$

$$\text{Also } \lim_{x \rightarrow +\infty} \frac{4^x}{2^x + 10} = \frac{\infty}{\infty} \text{ (L'H)} = \lim_{x \rightarrow +\infty} \frac{4^x \ln 4}{2^x \ln 2} = \lim_{x \rightarrow +\infty} 2^x \cdot \frac{\ln 4}{\ln 2} = +\infty$$

FOR 3).

Give an example of a sequence which is divergent and bounded.

$$\{1, 2, 1, 2, \dots\} \text{ or } \{(-1)^n\}$$

or any mixture of two convergent therefore bounded sequences:

$$a_n = \frac{1}{n} \text{ for } n \text{ odd and } a_n = \frac{n}{n+1} \text{ for } n \text{ even i.e. } \left\{1, \frac{2}{3}, \frac{1}{3}, \frac{4}{5}, \frac{1}{5}, \frac{6}{7}\right\}$$

so odd terms have limit 0, but even terms have limit 1,

so the whole sequence has NO limit.

D

FOR 1)

$$\lim_{n \rightarrow \infty} a_n = \lim_{x \rightarrow \infty} (x - 2^x) = \text{"}\infty - \infty\text{"}$$

so we have to compare, but we know that exp. function is stronger than any polynomial so limit is $-\infty$, to prove it:

$$\lim_{x \rightarrow \infty} 2^x \cdot \left(\frac{x}{2^x} - 1\right) = \lim_{x \rightarrow \infty} 2^x (L - 1),$$

where $L = \lim_{x \rightarrow \infty} \frac{x}{2^x}$ and we can use L'Hopital Rule since the type is $\frac{\infty}{\infty}$, so

$$L = \lim_{x \rightarrow \infty} \frac{1}{2^x \cdot \ln 2} = 0 \text{ since } \frac{1}{\infty} = 0. \text{ Together } \lim_{n \rightarrow \infty} a_n = +\infty \cdot (-1) = -\infty.$$

So the sequence is divergent and not bounded below. Is it bounded above?

Investigate: $a_1 = -1, a_2 = -3$, all terms are negative so an upper bound is 0, to see it compare the graphs $y = x$ and $y = 2^x$, the line is always below exp. function

Also $a_n \leq a_1 = -1$ since the sequence is decreasing: $a_{n+1} < a_n$ proof:

$$n + 1 - 2^{n+1} < n - 2^n \quad 1 < 2^n(2 - 1) \quad 1 < 2 \leq 2^n \text{ for all } n.$$

FOR 2). For $b_n = (n + 1)^{\frac{1}{n}}$ $b_n = f(n)$, where $f(x) = e^{\frac{1}{x} \ln(x+1)}$.

Calculate the limit of the exponent first:

$$L = \lim_{x \rightarrow \infty} \frac{\ln(x+1)}{x} = \text{"}\frac{\infty}{\infty}\text{" } L'H = \lim_{x \rightarrow \infty} \frac{\frac{1}{x+1}}{1} = 0, \text{ so } \lim_{n \rightarrow \infty} b_n = e^0 = 1$$

for monotonicity: $f'(x) = e^{\frac{1}{x} \ln(x+1)} \left[\frac{\frac{x}{x+1} - \ln(x+1)}{x^2} \right] < 0$

since $\frac{x}{x+1} - \ln(x+1) = \frac{x+1-1}{x+1} - \ln(x+1) = 1 - \frac{1}{x+1} - \ln(x+1) < 0$

if $1 < \ln(x+1)$ which is true for sure for $x \geq 2$

thus the sequence is decreasing;

also by definition $b_{n+1} < b_n$ $(n+2)^{\frac{1}{n+1}} < (n+1)^{\frac{1}{n}}$

$$(n+2)^n < (n+1)^{n+1} \quad \left(\frac{n+2}{n+1}\right)^n < n+1$$

$$\left(1 + \frac{1}{n+1}\right)^n < n+1 \text{ using the binomial formula}$$

$$\sum_{k=0}^{k=n} \binom{n}{k} \left(\frac{1}{n+1}\right)^k < \sum_{k=0}^{k=n} 1 = n+1$$

FOR 3)

For a) the sequence must be convergent e.g. $a_n = 1 - \frac{1}{n}$

and it is increasing since $\left\{\frac{1}{n}\right\}$ is decreasing

For b) $a_n = 3 + (-1)^n \frac{1}{n}$ since $\frac{1}{n} \rightarrow 0$, and also

the alternating sequence $\frac{(-1)^n}{n} \rightarrow 0$ by Squeeze Theorem